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Computation of the Taut, the Veering and the Teichmüller Polynomials

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ABSTRACT

Landry, Minsky and Taylor (LMT) introduced two polynomial invariants of veering triangulations—the taut polynomial and the veering polynomial. Here, we consider a pair of taut polynomials associated to one veering triangulation, the upper and the lower one, and analogously the upper and lower veering polynomials. We prove that the upper and lower taut polynomials are equal. In contrast, the upper and lower veering polynomials of the same veering triangulation may differ by more than a unit. We give algorithms to compute all these invariants. LMT related the Teichmüller polynomial of a fibered face of the Thurston norm ball with the taut polynomial of the associated layered veering triangulation. We use this result to give an algorithm to compute the Teichmüller polynomial of any fibered face of the Thurston norm ball.

KEYWORDS

3-manifolds; veering triangulation; taut polynomial; veering polynomial; Teichmüller polynomial; fibration over the circle

1. Introduction

Transverse taut veering triangulations were introduced by Ian Agol as a way to canonically triangulate certain pseudo-Anosov mapping tori [1]. More generally, they are tightly connected to pseudo-Anosov flows without perfect fits [19]. Here we study the *taut polynomial*, the *veering polynomial* and the *flow graph* of a (transverse taut) veering triangulation. All of them were introduced by Landry et al. [12]. We explicitly connect these invariants with the *upper track* of a veering triangulation; see Section 3.3. Since there is a “twin” train track, called the *lower track*, it is natural to distinguish the upper and lower taut polynomials, the upper and lower veering polynomials, and the upper and lower flow graphs. In the case of each invariant, we answer the question whether its lower and upper variants are the same up to some equivalence relation.

Proposition 5.2. *Let \mathcal{V} be a finite veering triangulation. The lower taut polynomial of \mathcal{V} is (up to a unit) equal to the upper taut polynomial of \mathcal{V} .*

Since the taut polynomial is generally well-defined only up to a unit, we conclude that there is only one taut polynomial associated to a veering triangulation. A similar statement does not hold for the flow graphs and the veering polynomials. We give examples which show the following.

Proposition 6.1. *There exists a veering triangulation \mathcal{V} whose upper and lower flow graphs are not isomorphic.*

Proposition 6.5. *There exists a veering triangulation \mathcal{V} whose upper and lower veering polynomials are not equal up to a unit.*

The majority of the results of this paper are computational. We simplify the original presentation for the (upper) taut module. As a result, the computation of the taut polynomial requires calculating only linearly many minors of a matrix, instead of exponentially many (Proposition 5.6). Using this we give algorithm `TautPolynomial` and prove that it correctly computes the taut polynomial of a veering triangulation in Proposition 5.8. Furthermore, we give algorithm `UpperVeeringPolynomial` and prove that it correctly computes the upper veering polynomial in Proposition 6.4. The lower veering polynomial of \mathcal{V} can be computed as the upper veering polynomial of the veering triangulation obtained from \mathcal{V} by reversing the coorientation on its 2-dimensional faces; see Remark 3.9.

McMullen introduced a polynomial invariant of fibered faces of the Thurston norm ball, called the *Teichmüller polynomial* [14, Section 3]. Its main feature is that it packages the stretch factors of monodromies of all fibrations lying over the fibered face [14, Theorem 5.1]. Landry, Minsky and Taylor proved that the taut polynomial of a *layered* veering triangulation \mathcal{V} is equal to the Teichmüller polynomial of the *fully-punctured* fibered face F determined by \mathcal{V} [12, Theorem 7.1]. Thus `TautPolynomial` can in particular be used to compute the Teichmüller polynomials of fully-punctured fibered faces. As observed in [12, Subsection 7.2] this can be generalized to all fibered faces via puncturing. We give two additional algorithms, `BoundaryCycles` and `Specialization`,

that together with `TautPolynomial` make up the algorithm `TeichmüllerPolynomial`. We then prove that this algorithm correctly computes the Teichmüller polynomial of any fibered face of the Thurston norm ball.

Proposition 8.6. *Let $\psi : S \rightarrow S$ be a pseudo-Anosov homeomorphism. Denote by N its mapping torus. Let F be the fibered face of the Thurston norm ball in $H_2(N, \partial N; \mathbb{R})$ such that $[S] \in \mathbb{R}_+ \cdot F$. Then the output of `TeichmüllerPolynomial`(ψ) is equal to the Teichmüller polynomial of F .*

Different algorithms to compute the Teichmüller polynomial were previously known. First of all, there is McMullen’s original algorithm [14, Section 3]. It relies on fixing a particular fibration lying in the fibered cone $\mathbb{R}_+ \cdot F$ and analysing a lift of the stable train track of the monodromy to the maximal free abelian cover of the 3-manifold. This algorithm is general, but hard to implement. There are simpler algorithms which work in some special cases. In [13] Lanneau and Valdez gave an algorithm to compute the Teichmüller polynomial of punctured disk bundles. Baik, Wu, Kim and Jo gave an algorithm to compute the Teichmüller polynomial of *odd-block surface* bundles [2]. In [4] Billet and Lechti covered the case of alternating-sign Coxeter links. The algorithm `TeichmüllerPolynomial` given in this article is general and in principle can be applied to any hyperbolic, orientable 3-manifold fibered over the circle.

All algorithms presented in this article have been implemented by the author, Saul Schleimer and Henry Segerman. The source codes are available at [9]. This implementation is based on their Veering Census and accompanying tools for computing with veering triangulations. Since there are 51,766 layered veering triangulations in the Veering Census, the implementation of the algorithm `TautPolynomial` alone expands the existing collection of computed Teichmüller polynomials from a couple of examples [2, 4, 13, 14] to almost 52 thousand of examples. Further generalization to fiber-parallel Dehn fillings of layered veering triangulations, given in the algorithm `TeichmüllerPolynomial`, yields even more easily computable examples.

2. Veering triangulations

In this section we define veering triangulations and prove lemmas about the combinatorics of veering triangulations which are used later in the paper.

2.1. Ideal triangulations of 3-manifolds

An *ideal triangulation* of an oriented 3-manifold M with torus cusps is a decomposition of M into ideal tetrahedra. We denote an ideal triangulation by $\mathcal{T} = (T, F, E)$, where T, F, E denote the set of tetrahedra, triangles (2-dimensional faces) and edges, respectively. Throughout the paper we assume that ideal simplices of \mathcal{T} are ordered and equipped with an orientation.

Every triangle of a triangulation has two *embeddings* into two, not necessarily distinct, tetrahedra. The number of (embeddings of) triangles attached to an edge is called the *degree* of this edge. An edge of degree d has d embeddings into triangles and d embeddings into tetrahedra.

By *edges of a triangle/tetrahedron* or *triangles of a tetrahedron* we mean embeddings of these ideal simplices into the boundary of a higher dimensional ideal simplex. Similarly, by *triangles/tetrahedra attached to an edge* we mean triangles/tetrahedra in which the edge is embedded, together with this embedding. When we claim that two lower dimensional simplices of a higher dimensional simplex are different we mean that at least their embeddings are different.

2.1.1. Cusped and truncated models

If we truncate the corners of ideal tetrahedra of \mathcal{T} we obtain a 3-manifold with toroidal boundary components. The interior of this manifold is homeomorphic to M . For notational simplicity, we also denote it by M . This is a slight abuse of notation, but it should not cause much confusion. We freely alternate between the *cusped* and the *truncated* models of M . The main advantage of the latter is that we can consider curves in the boundary ∂M .

2.2. Transverse taut triangulations

Let t be an ideal tetrahedron. Assume that a coorientation is assigned to each of its faces. We say that t is *transverse taut* if on two of its faces the coorientation points into t and on the other two it points out of t [10, Definition 1.2]. We call the pair of faces whose coorientations point out of t the *top faces* of t and the pair of faces whose coorientations point into t the *bottom faces* of t . We also say that t is *immediately below* its top faces and *immediately above* its bottom faces.

We encode a transverse taut structure on a tetrahedron by drawing it as a quadrilateral with two diagonals — one on top of the other; see Figure 1. Then the convention is that the coorientations on all faces point towards the reader. In other words, we look at the presented transverse taut tetrahedron *from above*.

The *top diagonal* is the common edge of the two top faces of t and the *bottom diagonal* is the common edge of the two bottom faces of t . The remaining four edges of a transverse taut tetrahedron are called its *equatorial edges*. Presenting a transverse taut tetrahedron

t as in Figure 1 endows it with an abstract assignment of angles from $\{0, \pi\}$ to its edges; the angle π is assigned to the diagonal edges of t and the angle 0 is assigned to all its equatorial edges. Such an assignment of angles is called a *taut structure* on t [10, Definition 1.1].

A triangulation $\mathcal{T} = (T, F, E)$ is *transverse taut* if

- every ideal triangle $f \in F$ is assigned a coorientation so that every ideal tetrahedron $t \in T$ is transverse taut,
- for every edge $e \in E$ the sum of angles of the underlying taut structure of \mathcal{T} , over all embeddings of e into tetrahedra, equals 2π [10, Definition 1.2].

This implies that if e is an edge of a transverse taut triangulation then the triangles attached to e are grouped into two *sides*, separated by a pair of π angles at e ; see Figure 2. Using a fixed orientation on e we can distinguish the *left side* of e from its *right side*. The transverse taut structure also allows us to identify a pair of the *lowermost* and a pair of the *uppermost* (relative to the coorientation) triangles attached to e . In Figure 2, triangles f_1, f'_1 are the lowermost and triangles f_3, f'_2 are the uppermost.

We say that a tetrahedron $t \in T$ is *immediately below* $e \in E$ if e is the top diagonal of t , and *immediately above* e , if e is the bottom diagonal of t . The remaining $\deg(e) - 2$ tetrahedra attached to e are called its *side tetrahedra*. Similarly as with triangles, we distinguish a pair of the *lowermost* and a pair of the *uppermost* side tetrahedra of e .

We denote a triangulation with a transverse taut structure by (\mathcal{T}, α) . Note that if M has a transverse taut triangulation (\mathcal{T}, α) then it also has a transverse taut triangulation obtained from (\mathcal{T}, α) by reversing coorientations on all faces of \mathcal{T} . We denote this triangulation by $(\mathcal{T}, -\alpha)$. These two triangulations have the same underlying taut structure, with tops and bottoms of tetrahedra swapped.

Remark. Triangulations as described above were introduced by Lackenby [11], where they are called *taut triangulations*.

2.3. Veering triangulations

A (transverse taut) *veering tetrahedron* is an oriented transverse taut tetrahedron whose edges are colored either red or blue, and the pattern of coloring on the equatorial edges is precisely as in Figure 1. There is no restriction on how the diagonal edges are colored; this is indicated by coloring them black.

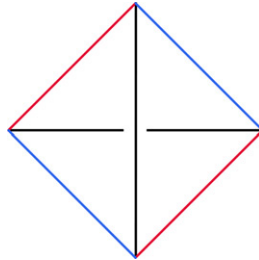


Figure 1. A veering tetrahedron. The underlying taut structure assigns the zero angle to the equatorial edges of the tetrahedron and the π angle to its diagonal edges.

More formally, we use [19, Definition 5.1] to define a veering tetrahedron as follows.

Definition 2.1 (Veering tetrahedron). Let t be a transverse taut tetrahedron, with all edges colored either red or blue. Then t is veering if the following two conditions hold.

- Let e_0, e_1, e_2 be edges of a top face of t , ordered counter-clockwise as viewed from above and so that e_0 is the top diagonal of t . Then e_1 is red and e_2 is blue.
- Let e_0, e_1, e_2 be edges of a bottom face of t , ordered counter-clockwise as viewed from above and so that e_0 is the bottom diagonal of t . Then e_1 is blue and e_2 is red.

A transverse taut triangulation is *veering* if a color (red/blue) is assigned to each edge of the triangulation so that every tetrahedron is veering [10, Definition 1.3]. By [10, Proposition 1.4], this definition is equivalent to Agol's original definition of a veering triangulation [1, Definition 4.1].

We denote a veering triangulation by $\mathcal{V} = (\mathcal{T}, \alpha, \nu)$, where ν corresponds to the coloring of edges. From a veering triangulation $(\mathcal{T}, \alpha, \nu)$ we can construct a veering triangulation $(\mathcal{T}, -\alpha, \nu)$, where the coorientations on faces are reversed, $(\mathcal{T}, \alpha, -\nu)$, where the colors on edges are interchanged and $(\mathcal{T}, -\alpha, -\nu)$, where both coorientations of faces and colors on edges are interchanged.

2.3.1. Colors of triangles of a veering triangulation

Note that any triangle of a veering triangulation has two edges of the same color and one edge of the other color; see Figure 1. This motivates the following definition.

Definition 2.2. We say that a triangle of a veering triangulation is *red* (respectively *blue*) if two of its edges are red (respectively blue).

In the following lemma and the subsequent corollary, we show how colors of triangles attached to an edge e depend on the color of e . We use this in the proof of [Proposition 6.4](#), where we prove that the algorithm `UpperVeeringPolynomial` correctly computes the upper veering polynomial.

Lemma 2.3. *Let \mathcal{V} be a veering triangulation. Then for any triangular face f the bottom diagonal of a tetrahedron immediately above f and the top diagonal of the tetrahedron immediately below f are distinct edges of f which have the same color.*

Proof. Let t be the tetrahedron immediately above f and let d be its bottom diagonal. Let t' be the tetrahedron immediately below f and let d' be its top diagonal. If d and d' are not distinct edges in the boundary of f , then the remaining edges of f cannot be colored so that the conditions from [Definition 2.1](#) are satisfied in both t and t' . Hence, d, d' are distinct edges in the boundary of f . Since d, d' are diagonal edges, there is another edge of f which has the same color as d and an edge of d' which has the same color as t . Thus, d, d' cannot be of a different color. \square

Note that [Lemma 2.3](#) in particular implies that every edge of a veering triangulation is of degree at least 4 and has at least two triangles on each of its sides.

Corollary 2.4. *Let \mathcal{V} be a veering triangulation. Among all triangles attached to an edge e , there are exactly four which have the same color as e . They are the two uppermost and the two lowermost triangles attached to e .*

Proof. By [Lemma 2.3](#), the lowermost and uppermost triangles attached to e are of the same color as e . Conversely, suppose f is neither a lowermost nor an uppermost triangle around e . Then e is an equatorial edge of both the tetrahedron t immediately above f and the tetrahedron t' immediately below f . Again by [Lemma 2.3](#), the bottom diagonal d of t and the top diagonal d' of t' are of the same color. The conditions from [Definition 2.1](#) imply that d, d' are distinct edges of f and thus f is of a different color than e . \square

2.3.2. The Veering Census

Data on transverse taut veering structures on ideal triangulations of orientable 3-manifolds consisting of up to 16 tetrahedra is available in the Veering Census [9]. A veering triangulation in the census is described by a string of the form

$$[\text{isoSig}]_{[\text{taut angle structure}]} \tag{2.5}$$

The first part of this string is the isomorphism signature of the triangulation. It identifies a triangulation uniquely up to combinatorial isomorphism. Isomorphism signatures have been introduced in [5, Section 3]. The second part of the string records the transverse taut structure, up to a sign.

The above description suggests that an entry from the Veering Census determines $([\mathcal{T}], \pm\alpha, \pm\nu)$, where $[\mathcal{T}]$ denotes the isomorphism class of \mathcal{T} . However, in the Veering Census certain (non-canonical) choices have been made. In fact, each entry corresponds to an ideal triangulation with numbered simplices, a fixed coorientation on its triangles, and a fixed orientation on its ideal simplices.

We use a string of the form (2.5) whenever we refer to a concrete example of a veering triangulation. Implementations of all algorithms given in this article take (2.5) as an input.

3. Structures associated to a transverse taut triangulation

In this section we recall the definitions of the *horizontal branched surface* [19, Subsection 2.12], the *boundary track* [8, Section 2], and the *upper* and *lower tracks* associated to a transverse taut triangulation [19, Definition 4.7]. The upper and lower tracks are closely related to the taut and veering polynomials; see [Sections 5](#) and [6](#). The boundary track is used in [Section 8.2](#) to encode boundary components of a surface carried by a transverse taut triangulation.

3.1. Horizontal branched surface

Let (\mathcal{T}, α) be a transverse taut triangulation of M . Since \mathcal{T} is endowed with a compatible taut structure, we can view the 2-skeleton $\mathcal{T}^{(2)}$ of \mathcal{T} as a 2-dimensional complex with a well-defined tangent space everywhere, including along its 1-skeleton. Thus $\mathcal{T}^{(2)}$ determines a *branched surface* (without vertices) in M [11, Introduction]. We call it the *horizontal branched surface* and denote it by \mathcal{B} [19, Subsection 2.12]. The branch locus of \mathcal{B} is equal to the 1-skeleton $\mathcal{T}^{(1)}$. In particular, we can see \mathcal{B} as *ideally triangulated* by the triangular faces of \mathcal{T} . We denote this triangulation of \mathcal{B} by (F, E) . For a more general definition of a branched surface, see Oertel [17, p. 532].

3.1.1. Branch equations

Let $e \in E$ be an oriented edge of degree d of a transverse taut triangulation \mathcal{T} . Let f_1, f_2, \dots, f_k be triangles attached to e on the left side, ordered from the bottom to the top. Let $f'_1, f'_2, \dots, f'_{d-k}$ be triangles attached to e on the right side, also ordered from the bottom to the top. Then e determines the following relation between the triangles attached to it:

$$f_1 + f_2 + \dots + f_k = f'_1 + f'_2 + \dots + f'_{d-k}. \quad (3.1)$$

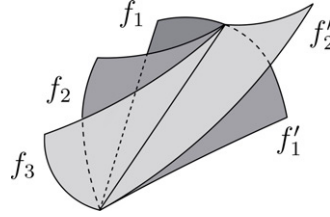


Figure 2. Edge with the branch equation $f_1 + f_2 + f_3 = f'_1 + f'_2$.

We call this equation the *branch equation* of e . An example is given in Figure 2. A transverse taut triangulation with n tetrahedra determines a system of n branch equations. In Section 4.1, we consider a matrix

$$B : \mathbb{Z}^E \rightarrow \mathbb{Z}^F, \quad (3.2)$$

which assigns to an edge e its branch equation. Matrix B is called the *branch equations matrix* for (\mathcal{T}, α) . For $e \in E$ as in (3.1) we set

$$B(e) = f_1 + f_2 + \dots + f_k - (f'_1 + f'_2 + \dots + f'_{d-k}).$$

3.1.2. Surfaces carried by a transverse taut triangulation

Let (\mathcal{T}, α) be a transverse taut triangulation. Let f_1, \dots, f_{2n} denote the triangular faces of \mathcal{T} . We say that a surface S properly embedded in M is *carried* by (\mathcal{T}, α) if there exists a nonzero, nonnegative, integral solution $w = (w_1, \dots, w_{2n}) \in \mathbb{Z}_{\geq 0}^{2n}$ to the system of branch equations of (\mathcal{T}, α) such that S can be realized up to isotopy by the relative 2-chain

$$S^w = \sum_{i=1}^{2n} w_i f_i.$$

In other words, S is carried by the horizontal branched surface of (\mathcal{T}, α) . More about the relationship between surfaces carried by a branched surface and weight systems on its sectors can be found in [20, Section II].

If there exists a strictly positive integral solution w to the system of branch equations of (\mathcal{T}, α) , then we say that (\mathcal{T}, α) is *layered*. In this case S^w is (a multiple of) a fiber of a fibration of M over the circle [12, Theorem 5.15].

3.2. Boundary track

An object which is strictly related to the horizontal branched surface is the *boundary track*. To define it, it is necessary to view the manifold M with a transverse taut triangulation (\mathcal{T}, α) in the truncated model. More information on the boundary track can be found in [8, Section 2].

Definition 3.3. Let (\mathcal{T}, α) be a (truncated) transverse taut triangulation of a (compact) 3-manifold M . Denote by \mathcal{B} the horizontal branched surface for (\mathcal{T}, α) . The *boundary track* β of (\mathcal{T}, α) is the intersection $\mathcal{B} \cap \partial M$.

In Figure 3, we present a local picture of a boundary track around one of its switches. A global picture of the boundary track of the veering triangulation `cPcbbbiht_12` of the figure eight knot complement is presented in Figure 13.



Figure 3. The boundary track around one of its switches. This switch corresponds to an endpoint of an edge of degree 7.

Each edge of \mathcal{T} has two endpoints. Therefore for every $e \in E$ the boundary track β has two switches of the same degree that can be labelled with e . Each triangle $f \in F$ has three arcs around its corners; see Figure 4. These corner arcs are in a bijective correspondence

with the branches of β . Therefore, for every $f \in F$, the track β has three branches that we label with f . If M has $b \geq 1$ boundary components T_1, \dots, T_b , then β is a disjoint union of train tracks β_1, \dots, β_b in boundary tori T_1, \dots, T_b , respectively.

The boundary track β of (\mathcal{T}, α) is transversely oriented by α . We orient the branches of β using the right hand rule and the coorientation on f ; see Figure 4. Therefore every switch has a collection of *incoming* branches and a collection of *outgoing* branches. Moreover, branches within these collections can be ordered from bottom to top. In particular, for every branch ϵ of β we can consider

- branches outgoing from the initial switch of ϵ above ϵ ,
- branches incoming to the terminal switch of ϵ above ϵ .

We use these observations in Section 8.2 where we give algorithm `BoundaryCycles`.

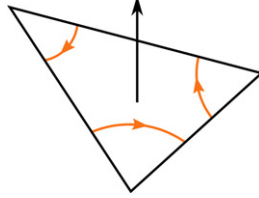


Figure 4. Coorientation on a triangle of \mathcal{T} determines orientation on the branches of β by the right-hand rule. Coorientation on the branches of β (not indicated in the picture) agrees with the coorientation on the triangle in which they are embedded.

3.3. Dual train tracks

In the previous subsection, we considered the boundary track associated to a transverse taut triangulation (\mathcal{T}, α) . In this subsection, we consider an entirely different kind of train tracks associated to (\mathcal{T}, α) , called *dual train tracks*. They are embedded in the horizontal branched surface \mathcal{B} and are dual to its triangulation (F, E) . A good reference for train tracks in surfaces is [18]. We need to modify the standard definition of a train track so that it is applicable to our setting.

We construct train tracks in \mathcal{B} dual to the triangulation (F, E) by gluing together “ordinary” train tracks in individual triangles of that triangulation. We restrict the class of train tracks that we allow in those triangles. The train tracks that we allow are called *triangular*.

Definition 3.4. Let f be an ideal triangle. By a *triangular train track* in f we mean a graph $\tau_f \subset f$ with four vertices and three edges, such that

- one vertex v is in the interior of f and the remaining three vertices are at the midpoints of the three edges in the boundary of f , one for each edge,
- for each vertex v' different than v there is an edge joining v and v' ,
- all edges are C^1 -embedded,
- there is a well-defined tangent line to τ_f at v .

See Figure 5. We call the vertex v in the interior of f a *switch* of τ_f . The edges of τ_f are called *half-branches*. Each half-branch has one *switch endpoint* and one *edge endpoint*.

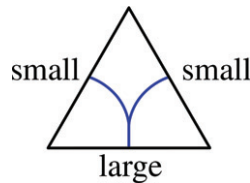


Figure 5. A triangular train track.

A tangent line to τ_f at a switch v distinguishes two *sides* of v . Two half-branches are on different sides of v if and only if the path contained in τ_f which joins their edge endpoints is smooth. A switch v has one half-branch on one side and two on the other. We call the half-branch which is the unique half-branch on one side of v the *large half-branch* of τ_f . The remaining two half-branches are called *small half-branches* of τ_f .

Definition 3.5. A *dual train track* in (\mathcal{B}, F) is a finite graph $\tau \subset \mathcal{B}$ whose restriction to any ideal triangle f of the ideal triangulation of \mathcal{B} by (F, E) is a triangular train track. We denote the restriction of τ to f by τ_f . Every *switch/half-branch* of τ is a *switch/half-branch* of τ_f for some $f \in F$, respectively.

3.3.1. The upper and lower tracks of a transverse taut triangulation

A transverse taut structure on a triangulation endows its horizontal branched surface \mathcal{B} with a pair of dual train tracks which we call, following [19, Definition 4.7], the *upper* and *lower* tracks of \mathcal{B} .

Definition 3.6. Let (\mathcal{T}, α) be a transverse taut triangulation. Let \mathcal{B} be the horizontal branched surface of (\mathcal{T}, α) equipped with the ideal triangulation (F, E) determined by \mathcal{T} . The *upper track* τ^U of \mathcal{T} is the dual train track in \mathcal{B} such that for every $f \in F$ the large-half branch of τ_f^U is dual to the bottom diagonal of the tetrahedron of \mathcal{T} immediately above f . The *lower track* τ^L of \mathcal{T} is the dual train track in \mathcal{B} such that for every $f \in F$ the large-half branch of τ_f^L is dual to the top diagonal of the tetrahedron of \mathcal{T} immediately below f .

We introduce the following names for the edges of $f \in F$ which are dual to large half-branches of τ_f^U or τ_f^L .

Definition 3.7. Let (\mathcal{T}, α) be a transverse taut triangulation. We say that an edge in the boundary of $f \in F$ is the *upper large* (respectively the *lower large*) edge of f if it contains the edge endpoint of the large half-branch of τ_f^U (respectively τ_f^L).

To define the upper and lower tracks, we do not need a veering structure on the triangulation. However, in the case of veering triangulations we can figure out the lower and upper tracks restricted to the faces of a given tetrahedron t without looking at the tetrahedra adjacent to t . Instead, the tracks are encoded by the colors of the edges of t ; see Figure 6. A more precise statement appears in the following lemma, which can be deduced from Lemma 3.2 of Landry et al. [12]. We use it in the proof of Lemma 5.4.

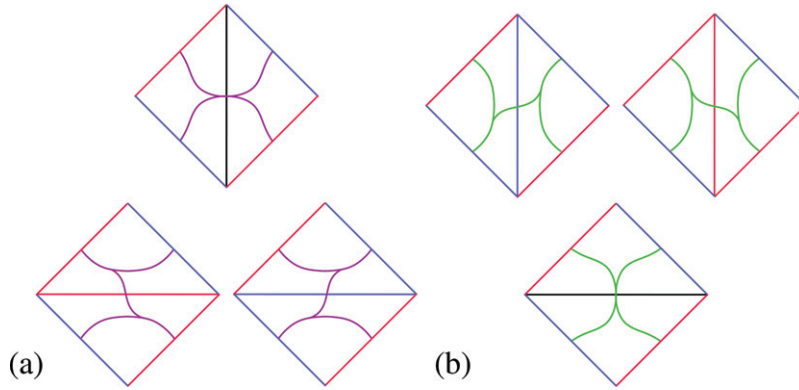


Figure 6. Squares in the top row represent top faces of a tetrahedron and squares in the bottom row represent bottom faces of a tetrahedron. (a) The lower track in a veering tetrahedron. In the bottom faces there are two options, depending on the color of the bottom diagonal. (b) The upper track in a veering tetrahedron. In the top faces there are two options, depending on the color of the top diagonal.

Lemma 3.8. Let \mathcal{V} be a veering triangulation. Let t be one of its tetrahedra. The upper large edges of the top faces of t are the equatorial edges of t which are of the same color as the top diagonal of t . The lower large edges of the bottom faces of t are the equatorial edges of t which are of the same color as the bottom diagonal of t . \square

The pictures of the lower and upper tracks in a veering tetrahedron are presented in Figure 6(a) and Figure 6(b), respectively.

Remark 3.9. The operation $v \mapsto -v$ does not affect the lower and upper tracks as their definition does not depend on the 2-coloring on the veering triangulation. The operation $\alpha \mapsto -\alpha$ interchanges the lower and upper track.

Remark 3.10. Landry, Minsky and Taylor work with the *stable branched surface* associated to \mathcal{V} [12, Section 4.2]. The upper track of \mathcal{V} is the intersection of this branched surface with the horizontal branched surface of \mathcal{V} . Similarly, the lower track of \mathcal{V} is the intersection of the *unstable branched surface* of \mathcal{V} [12, Section 5.2] with the horizontal branched surface of \mathcal{V} .

4. Free abelian covers of transverse taut triangulations

The aim of this paper is to give algorithms to compute the taut, the veering and the Teichmüller polynomials. All of them are derived from certain modules associated to the *maximal free abelian cover* M^{fab} of a 3-manifold M . This covering space corresponds to the kernel of the homomorphism

$$\pi_1(M) \rightarrow H_1(M; \mathbb{Z}) / \text{torsion}.$$

The deck group of the covering $M^{fab} \rightarrow M$ is isomorphic to

$$H = H_1(M; \mathbb{Z}) / \text{torsion}.$$

We will be more general and consider a free abelian quotient H' of $H_1(M; \mathbb{Z})$. This generalization is important in [Section 8](#), where the group H' will arise as the maximal free abelian quotient of the homology group $H_1(N; \mathbb{Z})$ of a Dehn filling N of M . Associated to H' there is an *intermediate* free abelian cover of M

$$M^{fab} \rightarrow M' \rightarrow M$$

with the deck group isomorphic to H' .

Let r denote the rank of H' . The integral group ring $\mathbb{Z}[H']$ on H' is isomorphic to the ring $\mathbb{Z}[u_1^{\pm 1}, \dots, u_r^{\pm 1}]$ of Laurent polynomials. If a basis (b_1, \dots, b_r) of H' is fixed then we choose the isomorphism to be $b_i \mapsto u_i$. Then an element $v = \sum_{i=1}^r v_i b_i \in H'$, $v_i \in \mathbb{Z}$, can be encoded by the Laurent monomial $u^v = u_1^{v_1} \cdots u_r^{v_r}$.

Remark. Throughout the paper we use the multiplicative convention for H' .

Suppose M is equipped with a transverse taut triangulation (\mathcal{T}, α) . A free abelian cover M' admits a triangulation \mathcal{T}' induced by \mathcal{T} via the covering map $M' \rightarrow M$. It is also transverse taut, as coorientations on triangular faces can be lifted from \mathcal{T} . If \mathcal{T} is additionally veering then so is \mathcal{T}' . In the next subsection we explain how to encode the infinite triangulation \mathcal{T}' using only finite data.

4.1. Encoding the triangulation of a free abelian cover

Let (\mathcal{T}, α) be a finite transverse taut triangulation of M with the set T of tetrahedra, the set F of faces and the set E of edges. Let H' be a free abelian quotient of $H_1(M; \mathbb{Z})$. In this subsection first we explain how to use the *dual graph* of \mathcal{T}' to fix a convenient fundamental domain for the action of H' on \mathcal{T}' . Then we describe our conventions for labelling ideal simplices of \mathcal{T}' with H' -coefficients. Finally, we define H' -pairings which together with the fundamental domain and our labelling convention completely encode \mathcal{T}' .

4.1.1. The dual 2-complex and the dual graph

Let \mathcal{D} be the 2-complex dual to \mathcal{T} . If \mathcal{T} consists of n tetrahedra, then \mathcal{D} has n vertices, each corresponding to some $t \in T$, $2n$ edges, each corresponding to a triangular face $f \in F$, and n two-cells, each corresponding to an edge $e \in E$. We abuse the notation slightly and denote the vertex of \mathcal{D} dual to tetrahedron t by t , the edge of \mathcal{D} dual to face f by f and the 2-cell of \mathcal{D} dual to edge e by e .

By Γ we denote the 1-skeleton of \mathcal{D} . We call Γ the *dual graph* of \mathcal{T} . We endow Γ with the “upwards” orientation on edges coming from the coorientation on their dual faces. We call any (simplicial) cycle in Γ a *dual cycle*. This is different from [\[12, Section 5\]](#), where by dual cycles the authors mean only positive cycles.

4.1.2. Fixing a fundamental domain

Let Υ be a spanning tree of Γ . If \mathcal{T} has n tetrahedra, then Υ has n vertices $t \in T$ and $n - 1$ edges. Let Γ' be the dual graph of \mathcal{T}' . Fix a lift $\tilde{\Upsilon}$ of Υ to Γ' . The lift $\tilde{\Upsilon}$ determines a fundamental domain \mathcal{F} for the action of H' on \mathcal{T}' built from:

- the interiors of all tetrahedra of \mathcal{T}' dual to vertices of $\tilde{\Upsilon}$,
- bottom diagonal edges of tetrahedra of \mathcal{T}' dual to vertices of $\tilde{\Upsilon}$,
- the interiors of triangles of \mathcal{T}' dual to the edges of Γ' which join two vertices of $\tilde{\Upsilon}$,
- the interiors of triangles of \mathcal{T}' dual to the edges of Γ' which run from a vertex not in $\tilde{\Upsilon}$ to a vertex of $\tilde{\Upsilon}$.

For our purposes it is sufficient to fix a fundamental domain up to a translation by an element $h \in H'$. That is, only the choice of a spanning tree is important and not its particular lift to Γ' . For this reason we say that \mathcal{F} constructed as above is the (downwardly closed) fundamental domain for the action of H' on \mathcal{T}' *determined by a spanning tree* Υ of Γ .

4.1.3. Labelling ideal simplices of the cover

Let \mathcal{F} be a fundamental domain for the action of H' on \mathcal{T}' determined by a spanning tree of Γ . Given an ideal simplex x of \mathcal{T} we denote its lift which is contained in \mathcal{F} by $1 \cdot x$. Every other lift of x is then a translate of $1 \cdot x$ by an element $h \in H'$; we denote it by $h \cdot x$. We say that $h \cdot x$ has the H' -coefficient h . We set

$$h \cdot \mathcal{F} = \left(\bigcup_{t \in T} h \cdot t \right) \cup \left(\bigcup_{f \in F} h \cdot f \right) \cup \left(\bigcup_{e \in E} h \cdot e \right).$$

In particular, $1 \cdot \mathcal{F} = \mathcal{F}$.

We denote the sets of ideal tetrahedra, triangles and edges of \mathcal{T}' by $H' \cdot T$, $H' \cdot F$, $H' \cdot E$, respectively. We set the action of H' on $H' \cdot X$, where $X \in \{T, F, E\}$, to be given by $h \cdot (g \cdot x) \mapsto (hg) \cdot x$. The above choices induce a particular identification of the free abelian groups generated by $H' \cdot T$, $H' \cdot F$ and $H' \cdot E$ with the free $\mathbb{Z}[H']$ -modules $\mathbb{Z}[H']^T$, $\mathbb{Z}[H']^F$, $\mathbb{Z}[H']^E$, respectively.

4.1.4. H' -pairings

To compute the taut and veering polynomials we will need to know the H' -coefficients of faces and tetrahedra attached to the edges of \mathcal{T}' with the H' -coefficient equal to 1; see proofs of [Propositions 5.8](#) and [6.4](#). By our fixed convention for labelling ideal simplices of \mathcal{T}' , the H' -coefficient of the tetrahedron immediately above $1 \cdot e$ is equal to 1. To figure out the H' -coefficients of the remaining tetrahedra adjacent to $1 \cdot e$ we introduce the notion of H' -pairings. These are elements of H' associated to the triangles of \mathcal{T} which inform about how much the labelling changes when traversing a lift of a given face in the upwards direction.

Definition 4.1. Let \mathcal{F} be a fundamental domain for the action of H' on \mathcal{T}' determined by a spanning tree Υ of Γ . The H' -pairings for $(\mathcal{T}', \mathcal{F})$ are elements $h_i \in H'$ associated to triangles $f_i \in F$ such that the tetrahedron immediately below $h \cdot f_i$ is in $h_i^{-1}h \cdot \mathcal{F}$. We also say that h_i is the H' -pairing of f_i (relative to \mathcal{F}).

Lemma 4.2. Suppose that an edge e is embedded in a tetrahedron t of \mathcal{T} . If e is the bottom diagonal of t , then in \mathcal{T}' the edge $1 \cdot e$ is the bottom diagonal of $1 \cdot t$. Otherwise let $f_{i_1} + \dots + f_{i_k}$ be a positive dual cycle from the vertex dual to t to a vertex dual to the tetrahedron immediately above e . Then the embedding of e into t induces an embedding of $1 \cdot e$ into $(h_{i_1} \dots h_{i_k})^{-1} \cdot t$. \square

Since \mathcal{F} is downwardly closed, the triangles of \mathcal{T}' inherit their H' -coefficient from the unique tetrahedron immediately above them. Therefore [Lemma 4.2](#) can also be used to find the H' -coefficients of triangles adjacent to $1 \cdot e$.

4.2. Finding H' -pairings

The dual 2-complex \mathcal{D} of \mathcal{T} is a deformation retract of M and therefore $H_1(M; \mathbb{Z}) \cong H_1(\mathcal{D}; \mathbb{Z})$. Hence $H_1(M; \mathbb{Z})$ is generated by dual cycles.

Given a collection C of dual cycles we consider the subgroup $\langle C \rangle$ generated by $\{[c] \in H_1(M; \mathbb{Z}) \mid c \in C\}$. Let

$$H_1(M; \mathbb{Z})^C = H_1(M; \mathbb{Z}) / \langle C \rangle$$

and set

$$H^C = H_1(M; \mathbb{Z})^C / \text{torsion}. \quad (4.3)$$

Every free abelian quotient H' of $H_1(M; \mathbb{Z})$ is equal to H^C for some finite collection C of dual cycles. Below we suppress C from notation, but we regularly use the notation H^C and \mathcal{T}^C later in the text.

Let Υ be a spanning tree of the dual graph Γ . Denote by F_Υ the subset of F consisting of triangles dual to the edges *not* in Υ . We call the elements of F_Υ the *non-tree edges*, and elements of $F - F_\Upsilon$ the *tree edges*. Recall that in [\(3.2\)](#) we associated to (\mathcal{T}, α) the branch equations matrix $B : \mathbb{Z}^E \rightarrow \mathbb{Z}^{F_\Upsilon}$. Let

$$B_\Upsilon : \mathbb{Z}^E \rightarrow \mathbb{Z}^{F_\Upsilon}$$

be the matrix obtained from B by deleting the rows corresponding to the tree edges.

Denote by \mathcal{D}_Υ the 2-complex obtained from \mathcal{D} by contracting Υ to a point. Then B_Υ is the boundary map from the 2-chains to the 1-chains of \mathcal{D}_Υ . Thus $H_1(\mathcal{D}_\Upsilon; \mathbb{Z})$, and hence $H_1(M; \mathbb{Z})$, is isomorphic to the cokernel of B_Υ . The maximal free abelian quotient H of $H_1(M; \mathbb{Z})$ is therefore isomorphic to

$$\langle F_\Upsilon \mid \{B_\Upsilon(e) \mid e \in E\} \rangle^{ab} / \text{torsion}.$$

Here the superscript ‘ab’ denotes the abelianization of the group presented in terms of generators and relations.

More generally, let C be a finite collection of dual cycles. Under the contraction $\mathcal{D} \rightarrow \mathcal{D}_\Upsilon$ they become simplicial 1-cycles in \mathcal{D}_Υ . We denote the obtained collection of cycles in \mathcal{D}_Υ by $C_\Upsilon = \{c_1, \dots, c_k\}$. The group $H' = H^C$ is isomorphic to

$$\langle F_\Upsilon \mid \{B_\Upsilon(e) \mid e \in E\}, C_\Upsilon \rangle^{ab} / \text{torsion}.$$

Suppose that H' is of rank r . Let $(B|C)_\Upsilon$ denote the matrix obtained from B_Υ by augmenting it with the columns c_1, \dots, c_k . Let

$$S = U(B|C)_\Upsilon V \quad (4.4)$$

be the Smith normal form of $(B|C)_\Upsilon$ (see [\[15, Chapter 1\]](#) for a discussion of Smith normal forms). Let f_1, \dots, f_{n+1} denote the elements of F_Υ . The matrix U transforms the basis (f_1, \dots, f_{n+1}) of \mathbb{Z}^{F_Υ} to another basis $(\mu_1, \dots, \mu_{n+1})$. The last r rows of both S and $U(B|C)_\Upsilon$ are zero, therefore $\{\mu_{n-r+2}, \dots, \mu_{n+1}\}$ are simplicial 1-cycles in \mathcal{D}_Υ whose images under the projection $\mathbb{Z}^{F_\Upsilon} \rightarrow H'$ form a basis of H' . For brevity, we say that $(\mu_{n-r+2}, \dots, \mu_{n+1})$ is a basis of H' .

The consecutive entries of the i -th column of U give the coefficients of f_i expressed as a linear combination of $(\mu_1, \dots, \mu_{n+1})$. Since $\mu_1, \dots, \mu_{n-r+1}$ are 0 in H' , it follows that the last r entries of the i -th column of U correspond to the H' -pairing of f_i written in terms of the basis $(\mu_{n-r+2}, \dots, \mu_{n+1})$ of H' .

On the other hand, the coefficients of μ_i expressed as a linear combination of (f_1, \dots, f_{n+1}) are equal to the consecutive entries of the i -th column of U^{-1} . Therefore the last r columns of the matrix U^{-1} give a representation of the basis elements of H' as simplicial 1-cycles in \mathcal{D}_Υ .

Remark 4.5. Adding a non-tree edge $f \in F_\Upsilon$ to the tree Υ creates a unique cycle $\zeta(f)$ in the subgraph $f \cup \Upsilon$ of Γ . The H' -pairing of f is equal to the image of the homology class of $\zeta(f)$ under the epimorphism $H_1(M; \mathbb{Z}) \rightarrow H'$.

Remark 4.6. The H' -pairings of $f \in F - F_\Upsilon$ are all trivial because tree edges correspond to contractible cycles contained in the tree Υ .

Remark. All computations presented in this section can be easily generalized to ideal triangulations which are not transverse taut. The only benefit of having a transverse taut structure is that we get a canonical choice of orientation on the edges of the dual graph.

4.3. Algorithm FacePairings

We give the algorithm `FacePairings` which lays a foundation for all other algorithms given in this paper. It performs computations discussed in Section 4.2 to find the projection $\mathbb{Z}^F \rightarrow H'$ which sends a dual cycle to the image of its homology class under $H_1(M; \mathbb{Z}) \rightarrow H'$. A free abelian quotient H' is specified by a finite collection C of dual cycles as in (4.3). The default output of `FacePairings` is a tuple of $2n$ Laurent monomials, which we call *face Laurents*. They encode the H' -pairings expressed with respect to the fixed basis of H' .

Algorithm FacePairings

Encoding the triangulation of a free abelian cover

Input:

- A transverse taut triangulation (\mathcal{T}, α) of a cusped 3-manifold M with n ideal tetrahedra, $\mathcal{T} = (T, F, E)$
- A list C of dual cycles of (\mathcal{T}, α)
- Optional: return type = “matrix”

Output:

- Default: a tuple of $2n$ Laurent monomials encoding the triangulation \mathcal{T}^C of a free abelian cover of M with the deck group isomorphic to H^C
- If return type = “matrix”: a pair (U, r) where r is the rank of H^C and U is as in (4.4)

```

1:  $B :=$  the branch equations matrix of  $(\mathcal{T}, \alpha)$  #  $2n \times n$  integer matrix
2:  $B := B.\text{AddColumns}(C)$ 
3:  $Y := \text{SpanningTree}(\mathcal{T})$ 
4:  $\text{NonTree} := F - Y$ 
5:  $B := B.\text{DeleteRows}(Y)$  #  $(n+1) \times n$  integer matrix
6:  $S, U, V := \text{SmithNormalForm}(B)$  #  $S = UB$ 
7:  $r :=$  the number of zero rows of  $S$ 
8: if return type = “matrix” then
9:   return  $U, r$ 
10: else
11:    $\mathcal{H} :=$  the zero matrix with  $r$  rows and columns indexed by elements of  $F$ 
12:   for  $f$  in  $\text{NonTree}$  do
13:     column  $\mathcal{H}(f) :=$  the last  $r$  entries of the column  $U(f)$ 
14:   end for
15:    $\text{FaceLaurents} :=$  the tuple of zero Laurent polynomials, indexed by  $F$ 
16:   for  $f$  in  $F$  do
17:      $\text{FaceLaurents}(f) := u^{\mathcal{H}(f)}$  #  $u = (u_1, \dots, u_r)$  and  $u^v = u_1^{v_1} \cdots u_r^{v_r}$ 
18:   end for
19:   return  $\text{FaceLaurents}$ 
20: end if

```

By `SpanningTree` we denote an algorithm which takes as an input an ideal triangulation $\mathcal{T} = (T, F, E)$, fixes a spanning tree Υ of its dual graph, and returns the subset of F consisting of triangles dual to the edges Υ . There are standard algorithms to find spanning trees of finite graphs, so we do not include pseudocode here.

Remark 4.7. We can ensure that the algorithm `FacePairings` is deterministic. That is, if we do not permute ideal simplices of \mathcal{T} , nor change the order of dual cycles in C , the output of `FacePairings` $((\mathcal{T}, \alpha), C)$ is always the same.

4.4. Polynomial invariants of finitely presented $\mathbb{Z}[H]$ -modules

Let \mathcal{M} be a finitely presented module over $\mathbb{Z}[H]$. Then there exist integers $k, l \in \mathbb{N}$ and an exact sequence

$$\mathbb{Z}[H]^k \xrightarrow{A} \mathbb{Z}[H]^l \longrightarrow \mathcal{M} \longrightarrow 0$$

of $\mathbb{Z}[H]$ -homomorphisms called a *free presentation* of \mathcal{M} . The matrix of A , written in terms of any bases of $\mathbb{Z}[H]^k$ and $\mathbb{Z}[H]^l$, is called a *presentation matrix* for \mathcal{M} .

Definition 4.8. [16, Section 3.1] Let \mathcal{M} be a finitely presented $\mathbb{Z}[H]$ -module with a presentation matrix A of dimension $l \times k$. We define the i -th *Fitting ideal* $\text{Fit}_i(\mathcal{M})$ of \mathcal{M} to be the ideal in $\mathbb{Z}[H]$ generated by $(l - i) \times (l - i)$ minors of A .

In particular $\text{Fit}_i(\mathcal{M}) = \mathbb{Z}[H]$ for $i \geq l$, as the determinant of the empty matrix equals 1, and $\text{Fit}_i(\mathcal{M}) = 0$ for $i < 0$ or $i < l - k$. The Fitting ideals are independent of the choice of a free presentation for \mathcal{M} [16, p. 58].

Remark. Fitting ideals are called *determinantal ideals* in [23] and *elementary ideals* in [6, Chapter VIII].

The ring $\mathbb{Z}[H]$ is a GCD domain [6, p. 117]. We define the i -th *Fitting invariant* of \mathcal{M} to be the greatest common divisor of elements of $\text{Fit}_i(\mathcal{M})$. When $\text{Fit}_i(\mathcal{M}) = (0)$ we set the i -th Fitting invariant of \mathcal{M} to be equal to 0. Note that Fitting invariants are well-defined only up to a unit in $\mathbb{Z}[H]$.

5. The taut polynomial

Let $\mathcal{V} = (\mathcal{T}, \alpha, \nu)$ be a finite veering triangulation of a 3-manifold M , with the set T of tetrahedra, the set F of 2-dimensional faces and the set E of edges. Recall that

$$H = H_1(M; \mathbb{Z}) / \text{torsion}.$$

In [12, Section 3], Landry, Minsky and Taylor defined the *face module* $\mathcal{E}_\alpha(\mathcal{V})$ of \mathcal{V} . They called the zeroth Fitting invariant of this module the *taut polynomial* of \mathcal{V} . In [12] the relation between the face module of \mathcal{V} and the upper track of \mathcal{V} is only implicit. Here we explicitly connect these two objects. For this reason, we denote the face module by $\mathcal{E}_\alpha^U(\mathcal{V})$ and call it the *upper taut module* of \mathcal{V} . Then the zeroth Fitting invariant of $\mathcal{E}_\alpha^U(\mathcal{V})$ is called the *upper taut polynomial*.

Let $\tau^{fab, U}$ be the upper track of the veering triangulation \mathcal{V}^{fab} of M^{fab} . Consider the restriction of $\tau^{fab, U}$ to $1 \cdot f \in H \cdot F$; see Figure 7. This train track determines a *switch relation* between its three half-branches: the large half-branch is equal to the sum of the two small half-branches. By identifying the half-branches with the edges in the boundary of $1 \cdot f$ which they meet, we obtain a switch relation between the edges in the boundary of $1 \cdot f$.

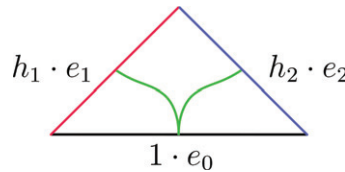


Figure 7. The upper track in a triangle determines the switch relation $1 \cdot e_0 = h_1 \cdot e_1 + h_2 \cdot e_2$ between the edges in its boundary.

The upper taut module $\mathcal{E}_\alpha^U(\mathcal{V})$ is generated over $\mathbb{Z}[H]$ by the edges of \mathcal{V} , with relations determined by the switch relations of $\tau^{fab, U}$. In other words, we can express $\mathcal{E}_\alpha^U(\mathcal{V})$ as the cokernel of the $\mathbb{Z}[H]$ -module homomorphism D^U

$$\mathbb{Z}[H]^F \xrightarrow{D^U} \mathbb{Z}[H]^E \longrightarrow \mathcal{E}_\alpha^U(\mathcal{V}) \longrightarrow 0, \quad (5.1)$$

where $D^U(1 \cdot f)$ is a rearrangement of the switch relation of $\tau^{fab, U}$ in $1 \cdot f$. For example, the image under D^U of the triangle presented in Figure 7 is equal to

$$1 \cdot e_0 - h_1 \cdot e_1 - h_2 \cdot e_2 \in \mathbb{Z}[H]^E.$$

The *upper taut polynomial* $\Theta_{\mathcal{V}}^U$ is the zeroth Fitting invariant of $\mathcal{E}_{\alpha}^U(\mathcal{V})$. That is,

$$\Theta_{\mathcal{V}}^U = \gcd \{ \text{maximal minors of } D^U \}.$$

Now let $\tau^{fab,L}$ be the lower track of the veering triangulation \mathcal{V}^{fab} of M^{fab} . There is a $\mathbb{Z}[H]$ -module homomorphism $D^L : \mathbb{Z}[H]^F \rightarrow \mathbb{Z}[H]^E$ which assigns to $1 \cdot f \in H \cdot F$ the switch relation of $\tau^{fab,L}$ in $1 \cdot f$. We define the *lower taut module* $\mathcal{E}_{\alpha}^L(\mathcal{V})$ of \mathcal{V} to be the cokernel of D^L , and the *lower taut polynomial* $\Theta_{\mathcal{V}}^L$ of \mathcal{V} to be the greatest common divisor of the maximal minors of D^L .

Remark. The subscript α in $\mathcal{E}_{\alpha}^U, \mathcal{E}_{\alpha}^L$ reflects the fact that to define these modules we just need a transverse taut structure α on the triangulation. This is because the upper and lower tracks exist in transverse taut triangulations even when they are not veering; see Definition 3.6. We, however, consider only the taut polynomials of veering triangulations. In our proofs we rely on the veering structure. For example, in Proposition 5.2 we use Lemma 2.3 and in Lemma 5.4 we use Lemma 3.8.

5.1. Only one taut polynomial

At the beginning of this section we defined two taut polynomials. In this section we prove that they are in fact equal up to a unit in $\mathbb{Z}[H]$.

Proposition 5.2. *Let \mathcal{V} be a finite veering triangulation. The lower taut module of \mathcal{V} is isomorphic to the upper taut module of \mathcal{V} . Hence*

$$\Theta_{\mathcal{V}}^L = \Theta_{\mathcal{V}}^U$$

up to a unit in $\mathbb{Z}[H]$.

Proof. Let $1 \cdot f \in H \cdot F$ be a red triangle of \mathcal{V}^{fab} . By Lemma 2.3 the tetrahedron immediately below $1 \cdot f$ has a red top diagonal t and the tetrahedron immediately above $1 \cdot f$ has a red bottom diagonal r , for some $t, r \in H \cdot E$. Furthermore, t and r are distinct edges in the boundary of $1 \cdot f$. We have

$$\begin{aligned} D^L(1 \cdot f) &= t - r - l \\ D^U(1 \cdot f) &= r - t - l \end{aligned}$$

for some $l \in H \cdot E$, so the signs of the two red edges of f are interchanged. A similar statement is true for blue triangles: the images of D^L and D^U on them differ by swapping the signs of the two blue edges.

If we multiply all columns of D^L corresponding to red triangles of \mathcal{V} by -1 , and all rows corresponding to blue edges by -1 , we obtain the matrix D^U . Hence the maximal minors of D^L and D^U differ at most by a sign. \square

Thus from now on we only write about the *taut polynomial* $\Theta_{\mathcal{V}}$ and the taut module $\mathcal{E}_{\alpha}(\mathcal{V})$.

Corollary 5.3. *The taut polynomials of (\mathcal{T}, α, v) , $(\mathcal{T}, -\alpha, v)$, $(\mathcal{T}, \alpha, -v)$ and $(\mathcal{T}, -\alpha, -v)$ are equal.*

Proof. This follows from Proposition 5.2 and Remark 3.9. \square

5.2. Reducing the number of relations

Suppose that \mathcal{V} consists of n tetrahedra. The original definition of the taut polynomial requires computing $\binom{2n}{n} > 2^n$ minors of D^U , which is an obstacle for efficient computation. However, the relations satisfied by the generators of the taut module are not linearly independent. In this subsection we give a recipe to systematically eliminate $n - 1$ relations.

The following lemma follows from [12, Lemma 3.2]. We include its proof, because it is important in Proposition 5.6.

Lemma 5.4. *Let \mathcal{V} be a veering triangulation. Each tetrahedron $t \in T$ induces a linear dependence between the columns of D^U corresponding to the triangles in the boundary of t .*

Proof. Suppose that $1 \cdot t$ has red equatorial edges r_1, r_2 , blue equatorial edges l_1, l_2 , bottom diagonal d_b and top diagonal d_t , where $r_1, r_2, l_1, l_2, d_b, d_t \in H \cdot E$. Such a tetrahedron is illustrated in Figure 8.

Let $f_1, f_2 \in H \cdot F$ be two bottom triangles of $1 \cdot t$ such that

$$\begin{aligned} D^U(f_1) &= d_b - r_1 - l_1 \\ D^U(f_2) &= d_b - r_2 - l_2. \end{aligned}$$

For $i = 1, 2$ denote by f'_i the top triangle of $1 \cdot t$ such that f'_i and f_i are adjacent in $1 \cdot t$ along the upper large edge of f'_i .

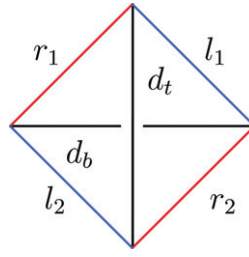


Figure 8. Edges of the tetrahedron $1 \cdot t$ of \mathcal{V}^{fab} .

By Lemma 3.8 the upper large edges of f'_i 's are the equatorial edges of $1 \cdot t$ which are of the same color as the top diagonal of $1 \cdot t$; see also Figure 6. Therefore

$$D^U(f_1 + f'_1) = D^U(f_2 + f'_2) = d_b - d_t - s_1 - s_2$$

$$\text{where } (s_1, s_2) = \begin{cases} (r_1, r_2) & \text{if } d_t \text{ is blue} \\ (l_1, l_2) & \text{if } d_t \text{ is red.} \end{cases}$$

□

Remark 5.5. Lemma 5.4 does not hold for transverse taut triangulations which do not admit a veering structure. One can check that if the upper large edges of the top faces of a tetrahedron t are not the opposite equatorial edges of t , then no nontrivial linear combination of the images of faces of $1 \cdot t$ under D^U gives zero.

Let Υ be a spanning tree of Γ . Recall that by F_Υ we denote the subset of triangles dual to the edges of Γ which are not in Υ (non-tree edges). We define a $\mathbb{Z}[H]$ -module homomorphism

$$D_\Upsilon^U : \mathbb{Z}[H]^{F_\Upsilon} \rightarrow \mathbb{Z}[H]^E$$

obtained from D^U by deleting the columns corresponding to the edges of Υ .

We will show that the images of D^U and that of D_Υ^U are equal. For brevity, we say that a dual edge f is a *linear combination of dual edges* f_{i_1}, \dots, f_{i_k} , or *in the span of these edges*, if $D^U(1 \cdot f)$ is a linear combination with $\mathbb{Z}[H]$ coefficients of $D^U(h_{i_1} \cdot f_{i_1}), \dots, D^U(h_{i_k} \cdot f_{i_k})$ for some $h_{i_1}, \dots, h_{i_k} \in H$.

Proposition 5.6. *Let \mathcal{V} be a finite veering triangulation and let Υ be a spanning tree of its dual graph Γ . The image of D^U and that of D_Υ^U are equal.*

Proof. It is enough to prove that every tree edge is in the span of non-tree edges. By Lemma 5.4 each dual edge f is a linear combination of three dual edges that share a vertex with f . In particular, the terminal edges of Υ — there are at least two of them — are in the span of non-tree edges. Now consider a subtree Υ' obtained from Υ by deleting its terminal edges. The terminal edges of Υ' can be expressed as linear combinations of non-tree edges and terminal edges of Υ , hence as linear combinations of non-tree edges only. Since Υ is finite, we eventually exhaust all its edges. □

Corollary 5.7. *Let Υ be a spanning tree of the dual graph Γ of a finite veering triangulation \mathcal{V} . The taut polynomial $\Theta_{\mathcal{V}}$ is equal to the greatest common divisor of the maximal minors of the matrix D_Υ^U .*

Proof. By Proposition 5.6 we obtain another presentation for the taut module

$$\mathbb{Z}[H]^{F_\Upsilon} \xrightarrow{D_\Upsilon^U} \mathbb{Z}[H]^E \rightarrow \mathcal{E}_\alpha(\mathcal{V}) \rightarrow 0.$$

Since Fitting invariants of a finitely presented module do not depend on a chosen presentation [16, p. 58], the greatest common divisor of the maximal minors of D_Υ^U is equal to the taut polynomial of \mathcal{V} . □

5.3. Algorithm TautPolynomial

In this subsection we present pseudocode for an algorithm which takes as an input a veering triangulation \mathcal{V} and outputs the taut polynomial $\Theta_{\mathcal{V}}$ of \mathcal{V} expressed in terms of the basis of H fixed by `FacePairings`(\mathcal{V} , []). To fill in the presentation matrix for the upper taut module we walk around the edges of \mathcal{V}^{fab} with the H -coefficient equal to 1 and record the H -coefficients of triangles attached to it.

In Section 7 we compute the taut polynomial of the veering triangulation `cPcbbbiht_12` of the figure-eight knot complement.

Algorithm TautPolynomial

Computation of the taut polynomial of a veering triangulation

Input: A veering triangulation \mathcal{V} with the set T of tetrahedra, the set F of triangular faces and the set E of edges
Output: The taut polynomial $\Theta_{\mathcal{V}}$

```

1: Pairing := FacePairings( $\mathcal{V}$ , [ ]) # Face Laurents encoding  $\mathcal{V}^{fab}$ 
2:  $D :=$  the zero matrix with rows indexed by  $E$  and columns by  $F$ 
3: for  $e$  in  $E$  do
4:    $L :=$  list of triangles on the left of  $e$ , ordered from the top to the bottom
5:    $R :=$  list of triangles on the right of  $e$ , ordered from the top to the bottom
6:   for  $A$  in  $\{L, R\}$  do
7:     CurrentCoefficient := 1 # Counting from 1, not 0
8:     add CurrentCoefficient to the entry  $(e, A[1])$  of  $D$ 
9:     for  $i$  from 2 to length( $A$ ) do # Inclusive
10:      CurrentCoefficient := CurrentCoefficient · Pairing( $A[i - 1]$ )
11:      subtract CurrentCoefficient from the entry  $(e, A[i])$  of  $D$ 
12:    end for
13:  end for
14: end for
15:  $Y :=$  SpanningTree( $Y$ )
16:  $D_Y := D.$ DeleteColumns( $Y$ ) # Accelerate the computation
17: minors :=  $D_Y.$ minors( $|E|$ )
18: return gcd(minors)

```

Proposition 5.8. *The output of TautPolynomial applied to a veering triangulation \mathcal{V} is equal to the taut polynomial $\Theta_{\mathcal{V}}$ of \mathcal{V} .*

Proof. The output FacePairings(\mathcal{V} , []) is a list of face Laurents encoding the triangulation \mathcal{V}^{fab} . Each for loop, starting on line 3 of the algorithm, is responsible for filling one row of the presentation matrix of the upper taut module.

By our conventions for labelling the triangles of \mathcal{V}^{fab} established in Section 4.1.3 the uppermost triangles attached to $1 \cdot e$ have the H -coefficient equal to 1. Then the H -coefficients of the consecutive (from the top) triangles attached to $1 \cdot e$ are obtained by multiplying the H -coefficient of the previous triangle by the inverse of its H -pairing; see Lemma 4.2. This explains line 10 of the algorithm. We do not invert H -pairings because if $h \cdot f$ is attached to $1 \cdot e$ then $1 \cdot f$ has $h^{-1} \cdot e$ in its boundary, and it is the latter we are interested in.

Since $1 \cdot e$ is the upper large edge only in the two uppermost triangles attached to $1 \cdot e$, we add the coefficients on line 8 and subtract on line 11.

Thus the matrix D on line 15 of the algorithm is equal to the presentation matrix D^U of the taut module $\mathcal{E}_{\alpha}(\mathcal{V})$ as in (5.1). Deleting the tree columns of D , for some spanning tree Υ of the dual graph Γ of \mathcal{V} , gives another presentation matrix for the taut module $\mathcal{E}_{\alpha}(\mathcal{V})$ by Corollary 5.7. The greatest common divisor of its maximal minors is equal to the zeroth Fitting invariant of $\mathcal{E}_{\alpha}(\mathcal{V})$, that is the taut polynomial $\Theta_{\mathcal{V}}$ of \mathcal{V} . \square

6. The veering polynomials

Let $\mathcal{V} = (\mathcal{T}, \alpha, \nu)$ be a finite veering triangulation of a 3-manifold M , with the set T of tetrahedra, the set F of 2-dimensional faces and the set E of edges. We still use the notation $H = H_1(M; \mathbb{Z}) / \text{torsion}$.

In Section 4 of Landry et al. [12], Landry, Minsky and Taylor defined the *flow graph* $\Phi_{\mathcal{V}}$ of \mathcal{V} . In Section 3 of the same paper they defined the *veering polynomial* $V_{\mathcal{V}}$ of \mathcal{V} . These two invariants are closely related: the veering polynomial is the image of the *Perron polynomial* of $\Phi_{\mathcal{V}}$ under the epimorphism induced by the inclusion of $\Phi_{\mathcal{V}}$ into M [12, Theorem 4.8]. Similarly as with the taut polynomial, here we explicitly connect $\Phi_{\mathcal{V}}$ and $V_{\mathcal{V}}$ with the upper track of \mathcal{V} . Thus we call them the *upper flow graph* and the *upper veering polynomial*, and denote them by $\Phi_{\mathcal{V}}^U$ and $V_{\mathcal{V}}^U$, respectively. Furthermore, using the lower track of \mathcal{V} we define the lower flow graph $\Phi_{\mathcal{V}}^L$, and the lower veering polynomial $V_{\mathcal{V}}^L$.

The aim of this section is twofold. First, we show examples of veering triangulations whose upper and lower flow graphs are not isomorphic (Proposition 6.1) and whose upper and lower veering polynomials are not equal in $\mathbb{Z}[H] / \pm H$ (Proposition 6.5). The second aim of this section is to present pseudocode for the computation of the upper veering polynomial (Section 6.3). By Remark 3.9, the lower veering polynomial of $(\mathcal{T}, \alpha, \nu)$ is equal to the upper veering polynomial of $(\mathcal{T}, -\alpha, \pm \nu)$, hence we do not give a separate pseudocode for its computation.

6.1. Flow graphs

The vertices of the *upper flow graph* Φ_V^U of \mathcal{V} are in bijective correspondence with the edges $e \in E$. Corresponding to each tetrahedron $t \in T$ there are three edges of Φ_V^U :

- from the bottom diagonal d_b of t to the top diagonal d_t of t ,
- from the bottom diagonal d_b of t to the equatorial edges s_1, s_2 of t which have a different color than the top diagonal of t .

This definition coincides with the definition of the flow graph given in [12, Subsection 4.3]. We additionally define the *lower flow graph* Φ_V^L of \mathcal{V} . Its vertices also correspond to the edges of the veering triangulation. Every tetrahedron $t \in T$ determines the following three edges of Φ_V^L :

- from the top diagonal d_t of t to the bottom diagonal d_b of t ,
- from the top diagonal d_t of t to the equatorial edges w_1, w_2 of t which have a different color than the bottom diagonal of t .

Observe that the edges of Φ_V^U (Φ_V^L) connect the upper (lower) large edge of the bottom (top) faces of t with the upper (lower) small edges of the top (bottom) faces of t .

Proposition 6.1. *There exists a veering triangulation \mathcal{V} whose upper and lower flow graphs are not isomorphic.*

Proof. The first entry of the Veering Census for which the upper and lower flow graphs are not isomorphic is given by the string hLMzMkbcdefggghhhqxqkc_1221002 which encodes a veering triangulation of the manifold v2898.

The graphs are presented in Figure 9. In 9(a) there are two vertices of valency 6 (labelled 4 and 6) which are joined to a vertex of valency 10 (labelled 0), while in 9(b) there is only one vertex of valency 6 (labelled 6) which is joined to a vertex of valency 10 (labelled 0). Hence the graphs are not isomorphic. \square

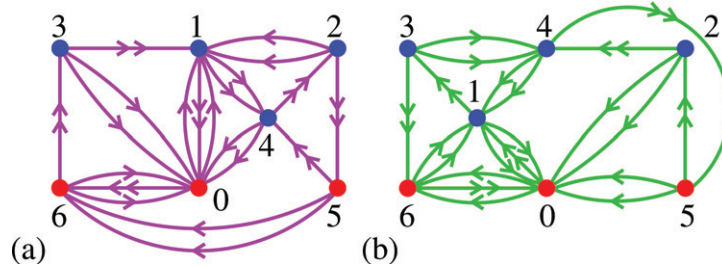


Figure 9. Flow graphs of hLMzMkbcdefggghhhqxqkc_1221002. Double arrows join top diagonals to bottom diagonals of tetrahedra, or vice versa.

6.2. Veering polynomials

The matrix D^U defined in (5.1) assigns to a face of \mathcal{V}^{fab} the switch relation of the upper track of \mathcal{V}^{fab} restricted to that face. By Lemma 5.4, the faces of a tetrahedron $h \cdot t$ of \mathcal{V}^{fab} can be grouped into pairs such that D^U evaluated on each pair equals

$$d_b - d_t - s_1 - s_2, \quad (6.2)$$

where d_t, d_b denote the top and the bottom diagonals of $h \cdot t$, respectively, and s_1, s_2 — its two equatorial edges of a different color than d_t . We view (6.2) as a relation associated to $h \cdot t$. By identifying $h \cdot t$ with its bottom diagonal we obtain a $\mathbb{Z}[H]$ -module homomorphism

$$\begin{aligned} K^U : \mathbb{Z}[H]^E &\rightarrow \mathbb{Z}[H]^E \\ K^U(d_b) &= d_b - d_t - s_1 - s_2. \end{aligned} \quad (6.3)$$

By choosing the same basis in the domain and codomain of K^U , the determinant of K^U is well-defined as an element of $\mathbb{Z}[H]$ (not just up to a unit). We call this polynomial the *upper veering polynomial* of \mathcal{V} and denote it by V_V^U . It was defined in [12, Section 3], where it is called the veering polynomial.

The map D^L , related to the lower track of \mathcal{V}^{fab} , satisfies analogous properties as D^U . In particular, we can group the triangles of $h \cdot t$ into pairs such that D^L evaluated on each pair equals $d_t - d_b - w_1 - w_2$, where w_1, w_2 denote the equatorial edges of $h \cdot t$ of a different color than d_b . By identifying $h \cdot t$ with its top diagonal we obtain a $\mathbb{Z}[H]$ -module homomorphism

$$\begin{aligned} K^L : \mathbb{Z}[H]^E &\rightarrow \mathbb{Z}[H]^E \\ K^L(d_t) &= d_t - d_b - w_1 - w_2. \end{aligned}$$

We call the determinant of K^L , when expressed with respect to the same basis in the domain and codomain, the *lower veering polynomial* of \mathcal{V} and denote it by $V_{\mathcal{V}}^L$.

6.3. Algorithm UpperVeeringPolynomial

In this section we present pseudocode for an algorithm which takes as an input a veering triangulation \mathcal{V} and returns its upper veering polynomial $V_{\mathcal{V}}^U$ expressed in terms of the basis of H fixed by $\text{FacePairings}(\mathcal{V}, [\])$. To fill in the matrix K^U we walk around the edges of \mathcal{V}^{fab} with the H -coefficient equal to 1 and record the H -coefficients of tetrahedra attached to it.

In Section 7 we compute the upper veering polynomial of the veering triangulation `cPcbbbiht_12` of the figure-eight knot complement.

Algorithm UpperVeeringPolynomial

Computation of the upper veering polynomial

Input: A veering triangulation \mathcal{V} , with the set T of tetrahedra, the set F of triangular faces and the set E of edges

Output: The upper veering polynomial $V_{\mathcal{V}}^U$ of \mathcal{V}

```

1: permute the elements of  $T$  so that  $E[i]$  is the bottom diagonal of  $T[i]$ 
2:  $\text{Pairing} := \text{FacePairings}(\mathcal{V}, [\ ])$  # Face Laurents encoding  $\mathcal{V}^{fab}$ 
3:  $K :=$  the zero matrix with rows indexed by  $E$  and columns by  $T$ 
4: for  $e$  in  $E$  do
5:    $L :=$  triangles on the left of  $e$ , ordered from the top to the bottom
6:    $R :=$  triangles on the right of  $e$ , ordered from the top to the bottom
7:    $TT :=$  tetrahedron immediately above  $L[1]$  # Counting from 1, not 0
8:   add 1 to the entry  $(e, TT)$  of  $K$ 
9:    $BT :=$  tetrahedron immediately below  $L[\text{length}(L)]$ 
10:   $\text{BottomCoefficient} := \prod_{i=1}^{\text{length}(L)} \text{Pairing}(L[i])$ 
11:  subtract  $\text{BottomCoefficient}$  from the entry  $(e, BT)$  of  $K$ 
12:  for  $A$  in  $\{L, R\}$  do
13:     $\text{CurrentCoefficient} := 1$ 
14:    for  $i$  from 1 to  $\text{length}(A)-1$  do # Inclusive, counting from 1, not 0
15:       $T :=$  tetrahedron immediately below  $A[i]$ 
16:       $\text{CurrentCoefficient} := \text{CurrentCoefficient} \cdot \text{Pairing}(A[i])$ 
17:      if  $i > 1$  then
18:        subtract  $\text{CurrentCoefficient}$  from the entry  $(e, T)$  of  $K$ 
19:      end if
20:    end for
21:  end for
22: end for
23: return determinant of  $K$ 

```

Proposition 6.4. *The output of UpperVeeringPolynomial applied to a veering triangulation \mathcal{V} is equal to the upper veering polynomial of \mathcal{V} .*

Proof. We claim that the matrix K on line 22 of the algorithm is equal to K^U defined in (6.3). Each for loop, starting on line 4 of the algorithm, is responsible for filling one row of K . By construction, K is a matrix of a $\mathbb{Z}[H]$ -module homomorphism $\mathbb{Z}[H]^T \rightarrow \mathbb{Z}[H]^E$. By our conventions for labelling ideal simplices of \mathcal{V}^{fab} established in Section 4.1.3, the bases for $\mathbb{Z}[H]^T \cong \mathbb{Z}[H]^E$ differ at most by a permutation (and not by multiplying by elements of H). Line 1 of the algorithm is therefore responsible for identifying a tetrahedron $h \cdot t$ with its bottom diagonal.

By our labelling convention, the tetrahedron immediately above $1 \cdot e$ have the H -coefficient equal to 1. Since $1 \cdot e$ is its bottom diagonal, we add 1 to the appropriate entry of K on line 8. On lines 10 and 16 we compute the H -coefficients of the remaining tetrahedra attached to $1 \cdot e$. We already explained this process in Lemma 4.2. Since $1 \cdot e$ is the top diagonal of the tetrahedron immediately below it, we subtract the coefficient computed on line 10 from the appropriate entry of K on line 11. Similarly, we subtract the coefficient computed on line 16 from the appropriate entry of K on line 18, because $1 \cdot e$ is an equatorial edge in all its side tetrahedra.

Line 17 of the algorithm is responsible for skipping the side tetrahedra of $1 \cdot e$ whose top diagonal has the same color as $1 \cdot e$. We know from Corollary 2.4 that only the two uppermost side tetrahedra of $1 \cdot e$ have this property. \square

An analogous algorithm can be written for the lower veering polynomial. Alternatively, by [Remark 3.9](#) to compute the upper veering polynomial of $(\mathcal{T}, \alpha, \nu)$ we can apply `UpperVeeringPolynomial` to the triangulation $(\mathcal{T}, -\alpha, \pm\nu)$.

Recall that the upper and lower veering polynomials are well-defined as elements of $\mathbb{Z}[H]$, and not just up to a unit. However, it only makes sense to compare them up to a unit. Using an implementation of `UpperVeeringPolynomial` we found that the upper and lower veering polynomials of the same veering triangulation may be different even after projecting them to $\mathbb{Z}[H] / \pm H$.

Proposition 6.5. *There are veering triangulations whose upper and lower veering polynomials project to different elements of $\mathbb{Z}[H] / \pm H$.*

Proof. The first entry of the Veering Census for which we have $V_{\mathcal{V}}^L \neq V_{\mathcal{V}}^U$ in $\mathbb{Z}[H] / \pm H$ is given by the string `iLLLAQccdfgfhhhggdatgqdm_21012210`. It encodes a veering triangulation of the 3-manifold `t10133`. Its upper and lower veering polynomials are respectively equal to

$$V_{\mathcal{V}}^U = u^{-57}(u-1)^{-1}(1-u^{29})(1-u^9) \\ (1-u+u^2-u^3+u^4-u^5+u^6)(1-u^2-u^7-u^{12}+u^{14})$$

and

$$V_{\mathcal{V}}^L = (u-1)^{-1}(1-u^{25})(1-u^{13}) \\ (1-u+u^2-u^3+u^4-u^5+u^6)(1-u^2-u^7-u^{12}+u^{14}).$$

Their greatest common divisor

$$(1-u+u^2-u^3+u^4-u^5+u^6)(1-u^2-u^7-u^{12}+u^{14})$$

is equal to the taut polynomial $\Theta_{\mathcal{V}}$ of \mathcal{V} . □

Remark 6.6. The flow graphs of the triangulation from the proof of [Proposition 6.5](#) are not isomorphic. In fact, one of them is planar, and the other is not.

Remark 6.7. In the proof of [Proposition 6.1](#) we showed that the upper and lower flow graphs of the veering triangulation `hLMzMkbcdefggghhhqxqkc_1221002` of the manifold `v2898` are not isomorphic. For this veering triangulation we have

$$V_{\mathcal{V}}^U = 1 - 19a - 19a^2 + a^3 = (1+a)(1-20a+a^2)$$

and

$$V_{\mathcal{V}}^L = a^{-3} - 19a^{-2} - 19a^{-1} + 1 = a^{-3}(1+a)(1-20a+a^2),$$

hence the veering polynomials are equal up to a unit.

There are even veering triangulations for which one veering polynomial vanishes and the other does not.

Example 6.8. The entry `lLLAPAMcbcfeggihijskshhxfpikaqj_20102220020` of the Veering Census encodes a veering triangulation whose upper veering polynomial vanishes, but

$$V_{\mathcal{V}}^L = u^{-1}(u-1)^2(u+1)^3(u^2-u+1)(u^4+1).$$

Remark 6.9. By the results of Landry, Minsky and Taylor the taut polynomial of \mathcal{V} divides the upper veering polynomial of \mathcal{V} [[12](#), Theorem 6.1 and Remark 6.18] and hence also the lower veering polynomial of \mathcal{V} . The remaining factors of the upper/lower veering polynomial are related to a special family of 1-cycles in the dual graph of \mathcal{V} , called the upper/lower *AB-cycles* [[12](#), Section 4]. We refer the reader to [[12](#), Subsection 6.1] to find out the formula for the extra factors.

If $\Theta_{\mathcal{V}} \neq 0$ and $V_{\mathcal{V}}^{U/L} = 0$, then \mathcal{V} has an upper/lower AB-cycle of even length whose class in H is trivial. From [Proposition 6.5](#) it follows that the homology classes of the lower and upper AB-cycles are not always paired so that one is the inverse of the other.

7. Example: the veering triangulation of the figure eight knot complement

Let M be the figure-eight knot complement. This 3-manifold admits a veering triangulation \mathcal{V} represented in the Veering Census by the string `cPcbbbiht_12`. In this section we compute the taut and upper veering polynomials of \mathcal{V} .

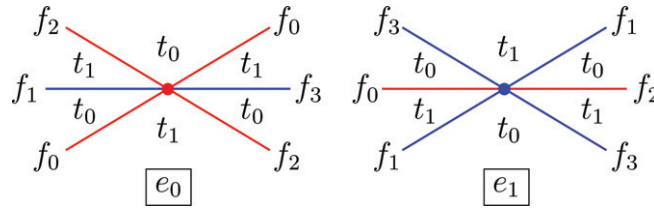


Figure 10. Cross-sections of the neighborhoods of edges e_0, e_1 of \mathcal{V} . The colors of edges and triangles are indicated.

7.1. Triangulation of the maximal free abelian cover

Recall from the proofs of Propositions 5.8 and 6.4 that we fill in the matrices used to compute the taut and veering polynomials by walking around the edges of \mathcal{V}^{fab} . For this reason, instead of presenting the tetrahedra of \mathcal{V} in Figure 10 we present cross-sections of the neighborhoods of its edges.

First we follow the algorithm FacePairings to encode the triangulation \mathcal{V}^{fab} of the maximal free abelian cover of the figure-eight knot complement. Figure 10 shows triangles attached to the edges e_0, e_1 of \mathcal{V} . It allows us to find the branch equations matrix B of \mathcal{V} :

$$B = \begin{matrix} & e_0 & e_1 \\ \begin{matrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{matrix} & \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & -1 \\ -1 & 0 \end{bmatrix} \end{matrix}.$$

Again using Figure 10 we draw the dual graph Γ of \mathcal{V} . It is presented in Figure 11. As a spanning tree Υ of Γ we choose $\{f_0\}$. The matrix B_Υ is obtained from B by deleting its first row, corresponding to f_0 . Let S be the Smith normal form of B_Υ . It satisfies

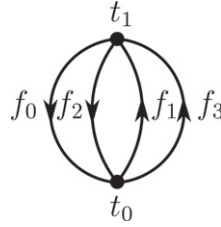


Figure 11. The dual graph of $cPcbbbt_{12}$.

$S = UB_\Upsilon V$, where

$$U = \begin{matrix} & f_1 & f_2 & f_3 \\ \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \end{matrix}.$$

Since S is of rank 2, $H_1(M; \mathbb{Z})$ is of rank 1, and thus the face Laurents of the non-tree edges are determined by the last row of U . All face Laurents for \mathcal{V} relative to the fundamental domain determined by $\Upsilon = \{f_0\}$ are listed in Table 1. This list is the output of FacePairings($\mathcal{V}, []$).

Table 1. The face Laurents encoding \mathcal{V}^{fab} .

face	f_0	f_1	f_2	f_3
face Laurent	1	u	1	u

Using Table 1 and Figure 10 we can now draw triangles and tetrahedra attached to the edges $1 \cdot e_0$ and $1 \cdot e_1$ of \mathcal{V}^{fab} in Figure 12.

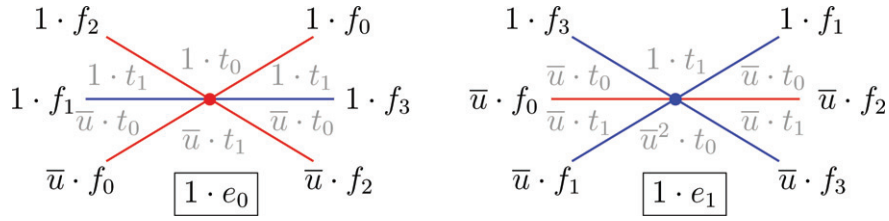


Figure 12. Cross-sections of the neighborhoods of edges $1 \cdot e_0$, $1 \cdot e_1$ of \mathcal{V}^{fab} . The colors of edges and triangles are indicated. To not overcrowd the figure we use $\bar{u} = u^{-1}$.

7.2. The taut polynomial

Recall from Section 5 that $D^U(1 \cdot f)$ is a linear combination of edges in the boundary of $1 \cdot f$. To find the matrix D^U we need to know the (inverses of) Laurent coefficients of the triangles attached to $1 \cdot e_0$ and $1 \cdot e_1$. They can be read off from Figure 12. Taking the inverses is necessary, because $1 \cdot f \in H \cdot F$ has $h \cdot e$ in its boundary if and only if $h^{-1} \cdot f$ is attached to $1 \cdot e$.

The upper large edge of $1 \cdot f$ appears with a plus sign in $D^U(1 \cdot f)$. The remaining edges of $1 \cdot f$ appear with a minus sign. This also can be read off from Figure 12, as $1 \cdot e_i$ is upper large only in its two uppermost triangles.

By Corollary 5.7 the taut polynomial of \mathcal{V} is equal to the greatest common divisor of the matrix D_Υ^U obtained from D^U by deleting its first column, corresponding to the tree $\Upsilon = \{f_0\}$. We have

$$D_\Upsilon^U(u) = \begin{matrix} & f_1 & f_2 & f_3 \\ \begin{matrix} e_0 \\ e_1 \end{matrix} & \begin{bmatrix} -1 & 1-u & -1 \\ 1-u & -u & 1-u \end{bmatrix} \end{matrix}$$

and hence

$$\Theta_{\mathcal{V}} = 1 - 3u + u^2.$$

7.3. The upper veering polynomial

Observe that $1 \cdot e_i$ is the bottom diagonal of $1 \cdot t_i$ for $i = 0, 1$. Therefore $K^U(1 \cdot e_i)$ is a linear combination of (not all) edges in the boundary of $1 \cdot t_i$. By Corollary 2.4 we skip the edges in the boundary of $1 \cdot t_i$ for which $1 \cdot t_i$ is an uppermost side tetrahedron. These edges have the same color as the top diagonal of $1 \cdot t_i$. Furthermore, only $1 \cdot e_i$ appear in $K^U(1 \cdot e_i)$ with a plus sign.

For simplicity, we view K^U as a map $\mathbb{Z}[H]^T \rightarrow \mathbb{Z}[H]^E$. Then it is clear that the row of K^U corresponding to e_i lists the inverses of Laurent coefficients of all but the two uppermost tetrahedra of $1 \cdot e_i$. These can be read off from Figure 12. We get

$$K^U(u) = \begin{matrix} & e_0 \sim t_0 & e_1 \sim t_1 \\ \begin{matrix} e_0 \\ e_1 \end{matrix} & \begin{bmatrix} 1-2u & -u \\ -u^2 & 1-2u \end{bmatrix} \end{matrix}.$$

Thus

$$V_{\mathcal{V}}^U = 1 - 4u + 4u^2 - u^3.$$

Up to a unit we have

$$V_{\mathcal{V}}^U = (u - 1) \cdot \Theta_{\mathcal{V}}.$$

8. The Teichmüller polynomial

Let N be a finite volume, oriented, hyperbolic 3-manifold. Thurston introduced a norm on $H_2(N, \partial N; \mathbb{R})$, now called the *Thurston norm*, whose unit ball $B_{\text{Th}}(N)$ is a polytope with rational vertices [21, Theorem 2]. He observed that if S is the fiber of a fibration of N over the circle then the homology class $[S]$ lies in the interior of the cone $\mathbb{R}_+ \cdot F$ on some top-dimensional face F of $B_{\text{Th}}(N)$. Moreover, in this case every primitive integral class $[S']$ from the interior of $\mathbb{R}_+ \cdot F$ can be represented by the fiber of a fibration of N over the circle [21, Theorem 3]. Top-dimensional faces of $B_{\text{Th}}(N)$ with the above property are called *fibred faces*.

Let F be a fibred face of the Thurston norm ball in $H_2(N, \partial N; \mathbb{R})$. Picking a primitive integral class from the interior of the cone $\mathbb{R}_+ \cdot F$ yields an expression of N as the mapping torus

$$N = (S \times [0, 1]) / \{(x, 1) \sim (\psi(x), 0)\}$$

of a pseudo-Anosov homeomorphism $\psi : S \rightarrow S$ [22, Proposition 2.6]. This homeomorphism is called the *monodromy of the fibration* $S \rightarrow N \rightarrow S^1$. Different fibrations lying over F have different pseudo-Anosov monodromies, with different *stretch factors*. All of them can be, however, encoded by a single polynomial invariant, called the *Teichmüller polynomial* of F and denoted by Θ_F [14, Theorem 5.1].

By a result of Agol [1, Section 4], there is a layered veering triangulation associated to every fibered face. To explain this, let us consider a fibration of N over the circle with fiber S and monodromy ψ . It determines the *suspension flow* on N defined as the unit speed flow along the curves $\{x\} \times [0, 1]$. This flow admits a finite number of closed *singular orbits* $\ell_1, \dots, \ell_k \subset N$. The singular orbits arise from the prong-singularities of the invariant foliations of ψ in S . Following Agol's algorithm [1, Section 4] yields a *layered veering triangulation* of $M = N - \{\ell_1, \dots, \ell_k\}$. Furthermore, any fibration from the cone $\mathbb{R}_+ \cdot F$ over the same fibered face determines the same veering triangulation. This follows from a result of Fried that (up to isotopy and reparametrization) the suspension flow does not depend on the chosen integral homology class in $\mathbb{R}_+ \cdot F$ [7, Theorem 14.11]. The veering triangulation is in fact an invariant of this flow.

Landry, Minsky and Taylor observed that the Teichmüller polynomial of a fibered face can be computed using the taut polynomial of the associated veering triangulation. Before we state their theorem, we set

$$\text{sing}(F) = \{\ell_1, \dots, \ell_k\}.$$

We also change the previous notation and set

$$H_N = H_1(N; \mathbb{Z}) / \text{torsion}, \quad H_M = H_1(M; \mathbb{Z}) / \text{torsion}.$$

Theorem 8.1 (Proposition 7.2 of Landry et al. [12]). *Let N be a compact, oriented, hyperbolic 3-manifold which is fibered over the circle. Let F be a fibered face of the Thurston norm ball in $H^1(N; \mathbb{R})$. Denote by \mathcal{V} the veering triangulation of $M = N - \text{sing}(F)$ associated to F . Let $i_* : H_M \rightarrow H_N$ be the epimorphism induced by the inclusion of M into N . Then*

$$\Theta_F = i_*(\Theta_{\mathcal{V}}).$$

□

In particular, if $\text{sing}(F) = \emptyset$, then $\Theta_F = \Theta_{\mathcal{V}}$. Fibered faces with this property are called *fully-punctured*. We conclude that the output of the algorithm `TautPolynomial` applied to a layered veering triangulation is equal to the Teichmüller polynomial of a fully-punctured fibered face. If F is not fully-punctured, there are two additional steps to compute Θ_F .

1. Finding Dehn filling slopes on the boundary tori of M which recover N .
2. Computing the specialisation of the taut polynomial of \mathcal{V} under $i_* : H_M \rightarrow H_N$.

In this section we give two algorithms, `BoundaryCycles` and `Specialization`, which realize the first and the second step, respectively. Then we give the algorithm `TeichmüllerPolynomial`, which relies on algorithms `BoundaryCycles`, `TautPolynomial` and `Specialization`, to compute the Teichmüller polynomial of any fibered face.

8.1. Classical (fully-punctured) examples

A majority of computations of the Teichmüller polynomials previously known in the literature concern only fully-punctured fibered faces. Such Teichmüller polynomials can be computed using the algorithm `TautPolynomial`. Table 2 compares the outputs of this algorithm with computations of other authors.

8.2. Algorithm `BoundaryCycles`

Let (\mathcal{T}, α) be a finite transverse taut triangulation of M with the set T of tetrahedra, the set F of faces and the set E of edges. Denote by f_1, \dots, f_{2n} the triangular faces of \mathcal{T} . Let $w = (w_1, \dots, w_{2n})$ be a nonzero, nonnegative, integral solution to the system of branch equations of (\mathcal{T}, α) . This solution determines a surface $S^w = \sum_{i=1}^{2n} w_i f_i$ in a fixed carried position.

The goal of this section is to present an algorithm which given (\mathcal{T}, α) and w as above outputs a collection C of dual cycles which are homologous to the boundary components of S^w .

Suppose that the truncated model of M has b boundary components T_1, \dots, T_b . The intersection $\partial S^w \cap T_j$ might be disconnected. In this case the dual cycle that we obtain is homologous to a *multiple* of a Dehn filling slope on T_j . Finding multiples of Dehn filling slopes is sufficient for our purpose, that is finding the projection $H_M \rightarrow H_M^C$, because H_M^C is by definition torsion-free; see (4.3).

The boundary components of the surface S^w are carried by the boundary track (defined in Section 3.2). The tuple w endows each boundary track β_1, \dots, β_b with a nonnegative integral transverse measure which encodes the boundary components $\partial S^w \cap T_j$ for $1 \leq j \leq b$.

The general idea to find a dual cycle c_j homologous to $\partial S^w \cap T_j$ is as follows.

Table 2. Teichmüller polynomials of fully-punctured fibered faces previously known in the literature compared to the outputs of `TautPolynomial` on the associated veering triangulations.

Example 1	
Source of the example	McMullen [14, Subsection 11.I]
Polynomial in the source	$1 - u - ut - ut^{-1} + u^2$
Veering triangulation \mathcal{V}	eLMkbcdddddde_2100
<code>TautPolynomial</code> (\mathcal{V})	$a^2b - a^2 - ab - b^2 + b$
Change of basis	$t \mapsto ab^{-1}, u \mapsto a$
Example 2	
Source of the example	McMullen [14, Subsection 11.II]
Polynomial in the source	$t^{-2} - ut - u - ut^{-1} - ut^{-2} - ut^{-3} + u^2$
Veering triangulation \mathcal{V}	ivvPQQcfghghfghfaddddaaaa_20000222
<code>TautPolynomial</code> (\mathcal{V})	$ab^4 - a^2b^2 + ab^3 + ab^2 + ab - b^2 + a$
Change of basis	$t \mapsto b^{-1}, u \mapsto ab$
Example 3	
Source of the example	Lanneau & Valdez [13, Subsection 7.2]
Polynomial in the source	$u^2 - ut_A t_B - ut_B - u - ut_A^{-1} + t_B$
Veering triangulation \mathcal{V}	gvLQQcdeffefffffaafa_201102
<code>TautPolynomial</code> (\mathcal{V})	$a^2bc^2 - abc - ac^2 - ab - ac + 1$
Change of basis	$t_A \mapsto bc^{-1}, t_B \mapsto b^{-1}c^2, u \mapsto ac^2$

1. Perturb $\partial S^w \cap T_j$ slightly, so that it becomes transverse to the boundary track.
2. Push the (perturbed) $\partial S^w \cap T_j$ away from the boundary of M into the dual graph Γ .

First we define an auxiliary object, the *dual boundary graph* Γ^β .

Definition 8.2. Let (\mathcal{T}, α) be a (truncated) transverse taut ideal triangulation of a 3-manifold M . The *dual boundary graph* Γ^β is the oriented graph contained in ∂M which is dual to the boundary track β of (\mathcal{T}, α) . The orientation on the edges of Γ^β is determined by α .

If $b \geq 1$ then the dual boundary graph is disconnected, with connected components $\Gamma_1^\beta, \dots, \Gamma_b^\beta$ such that Γ_j^β is dual to the boundary track β_j . If an edge of Γ^β is dual to a branch of β lying in $f \in F$, then we label it with f . Hence for every $f \in F$ there are three edges of Γ^β labelled with f .

Example 8.3. The dual boundary graph of the veering triangulation `cPcbbbiht_12` of the figure-eight knot complement is presented in Figure 13.

The dual boundary graph is a combinatorial tool that we use to encode paths which are transverse to the boundary track. Moreover, every cycle in the dual boundary graph can be homotoped inside M to a cycle in the dual graph.

Lemma 8.4. Let (\mathcal{T}, α) be a transverse taut triangulation of a 3-manifold M . Denote by Γ, Γ^β its oriented dual graph and its oriented dual boundary graph, respectively. Let c^β be a cycle in Γ^β . Suppose it passes consecutively through the edges of Γ^β labelled with f_{i_1}, \dots, f_{i_n} , where $1 \leq i_j \leq 2n$.

We set

$$s_{ij} = \begin{cases} +1 & \text{if } c \text{ passes through an edge labelled with } f_{i_j} \text{ upwards} \\ -1 & \text{if } c \text{ passes through an edge labelled with } f_{i_j} \text{ downwards.} \end{cases}$$

Let c be the cycle $(s_{i_1}f_{i_1}, \dots, s_{i_n}f_{i_n})$ in the dual graph Γ . If we embed Γ^β and Γ in M in the natural way, then c^β and c are homotopic.

Proof. Pushing each edge of the cycle c^β towards the middle of the triangle through which it passes gives a homotopy between c^β and c . This is illustrated in Figure 14. \square

Fix an integer j between 1 and b . The curve $\partial S^w \cap T_j$ is contained in the boundary track β_j . Let ϵ be a branch of β_j . Let s^- and s^+ be the initial and the terminal switches of ϵ , respectively. We replace each subarc of $\partial S^w \cap T_j$ contained in ϵ by the following 1-chain c_ϵ in Γ_j^β

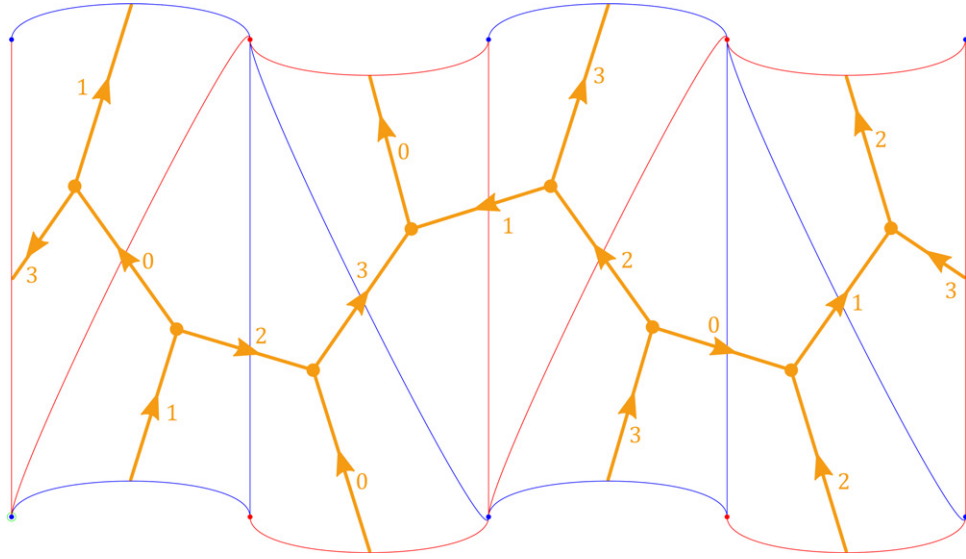


Figure 13. The boundary track β of the veering triangulation `cPcbbbiht_12` of the figure-eight knot complement and its dual graph Γ^β (in orange). The orientation on the edges of Γ^β is determined by the coorientation on their dual branches of β . Edges of Γ^β are labelled with indices of triangles that they pass through. The picture of the boundary track is taken from [9]. The dual boundary graph has been added by the author.

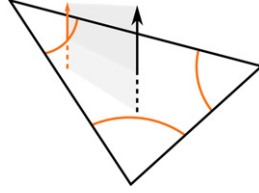


Figure 14. Homotoping a dual boundary cycle to a dual cycle. The black arrow is an edge of Γ dual to f . The orange arrow is an edge of Γ^β labelled with f .

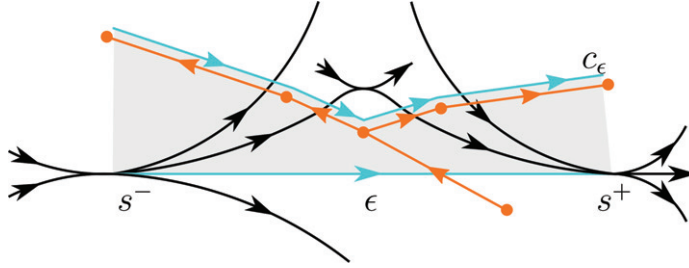


Figure 15. A local picture of the boundary track β (in black) and the dual boundary graph Γ^β (in orange). We push the branch ϵ of β (light blue) upwards to the 1-chain c_ϵ in Γ^β (also light blue).

$$c_\epsilon = - \left(\begin{array}{c} \text{outgoing branches} \\ \text{of } s^- \text{ above } \epsilon \end{array} \right) + \left(\begin{array}{c} \text{incoming branches} \\ \text{of } s^+ \text{ above } \epsilon \end{array} \right).$$

This is schematically depicted in Figure 15.

Let us denote the transverse measure on ϵ determined by $w = (w_1, \dots, w_{2n})$ by $w(\epsilon)$. The curve $\partial S^w \cap T_j$ passes through ϵ $w(\epsilon)$ times. Since chain groups are abelian, the 1-cycle in Γ^β homologous to $\partial S^w \cap T_j$ is given by

$$c_j^\beta = \sum_{\epsilon \in \beta_j} w(\epsilon) c_\epsilon,$$

where the sum is taken over all branches ϵ of β_j . By Lemma 8.4 we can homotope the cycles c_j^β in Γ^β to cycles c_j in Γ .

The procedure explained in this section is summed up in the algorithm `Boundary Cycles` below. In the algorithm we use the notion of *upward* and *downward* edges. They are defined as follows. A vertex v of an ideal triangle $f \in F$ gives a branch ϵ_v of β . We say that an edge e of f is the *downward* edge for v in f if its intersection with ∂M is the initial switch of ϵ_v . An edge e of f is the

Algorithm BoundaryCyclesExpressing boundary components of a surface carried by a transverse taut triangulation (\mathcal{T}, α) as dual cycles**Input:**

- A transverse taut triangulation (\mathcal{T}, α) with n tetrahedra and b ideal vertices, $\mathcal{T} = (T, F, E)$
- A nonzero tuple $w \in \mathbb{Z}^{2n}$ of integral nonnegative weights on elements of F

Output:

- A list of b vectors from \mathbb{Z}^{2n} , each encoding a dual cycle c_j homotopic to $\partial S^w \cap T_j$, for $1 \leq j \leq b$

```

1: Boundaries := the list of  $b$  zero vectors from  $\mathbb{Z}^{2n}$ 
2: for  $f$  in  $F$  do
3:   for vertex  $v$  of  $f$  do
4:      $j :=$  the index of  $v$  as an ideal vertex of  $\mathcal{T}$ 
5:      $e_1, e_2 :=$  the downward and upward edges of  $v$  in  $f$ 
6:     for  $f'$  above  $f$  on the same side of  $e_1$  do
7:       subtract  $w(f)$  from the entry  $f'$  of Boundaries[ $j$ ]
8:     end for
9:     for  $f'$  above  $f$  on the same side of  $e_2$  do
10:      add  $w(f)$  to the entry  $f'$  of Boundaries[ $j$ ]
11:    end for
12:  end for
13: end for
14: return Boundaries

```

upward edge for v in f if its intersection with ∂M is the terminal switch of ϵ_v . The names reflect the fact that when we homotope the branch ϵ_v to a 1-chain in Γ^β we go downwards above the initial switch of ϵ_v and upwards above the terminal switch of ϵ_v ; see Figure 15.

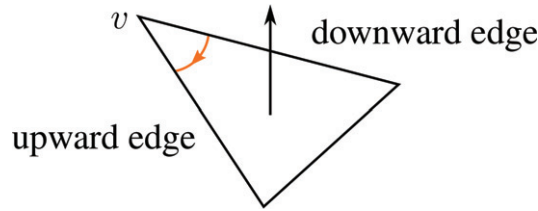


Figure 16. Downward and upward edges for an ideal vertex v of a triangle.

Remark. Algorithm BoundaryCycles is due to Saul Schleimer and Henry Segerman. We include it here, with permission, for completeness.

8.3. Algorithm Specialization

Let (\mathcal{T}, α) be a finite transverse taut triangulation of a 3-manifold M . Let C be a finite collection of dual cycles of (\mathcal{T}, α) . It determines an epimorphism $\rho_C : H_M \rightarrow H_M^C$, where the meaning of the superscript C is as in (4.3). In this section we give an algorithm to compute $\rho_C(P)$ given $P \in \mathbb{Z}[H_M]$.

Observe that the epimorphism ρ_C is a generalization of the epimorphism $i_* : H_M \rightarrow H_N$ induced by the inclusion of M into its Dehn filling N . Indeed, to express i_* as ρ_C it is enough to find a collection C of dual cycles which are homologous to the Dehn filling slopes which produce N from M . In the previous section we explained how to find C in the special case when the Dehn filling is determined by the boundary components of a surface carried by (\mathcal{T}, α) .

Algorithm SpecializationComputing the specialisation of $P \in \mathbb{Z}[H_M]$ under the epimorphism determined by a collection of dual cycles**Input:**

- A veering triangulation \mathcal{V} of a 3-manifold M
- $P \in \mathbb{Z}[H_M]$ expressed in terms of the basis fixed by $\text{FacePairings}(\mathcal{V}, [\])$
- A finite list C of dual cycles of \mathcal{V}

Output: The specialisation $\rho_C(P)$ of P

```

1:  $n :=$  the number of tetrahedra of  $\mathcal{V}$ 
2:  $U, r := \text{FacePairings}(\mathcal{V}, [\ ])$ , return type = “matrix”
3:  $A_1 :=$  the matrix obtained from  $U^{-1}$  by deleting its first  $n + 1 - r$  columns
4:  $U', s := \text{FacePairings}(\mathcal{V}, C)$ , return type = “matrix”
5:  $A_2 :=$  the matrix obtained from  $U'$  by deleting its first  $n + 1 - s$  rows
6:  $\text{exp} := \text{Exponents}(P)$ 
7:  $\text{coeff} := \text{Coefficients}(P)$ 
8:  $\text{newExp} := [\ ]$ 
9: for  $v$  in  $\text{exp}$  do
10:   append  $\text{newExp}$  with  $A_2 A_1(v)$ 
11: end for
12:  $\text{spec} := \sum_{i=1}^{\text{length}(\text{coeff})} \text{coeff}[i] \cdot u^{\text{newExp}[i]}$ 
13: return  $\text{spec}$ 

```

$\# u = (u_1, \dots, u_s)$ and $u^v = u_1^{v_1} \cdots u_s^{v_s}$

Proposition 8.5. Let (\mathcal{T}, α) be a finite transverse taut triangulation of a 3-manifold M . Let $P \in \mathbb{Z}[H_M]$ be expressed in terms of the basis fixed by $\text{FacePairings}((\mathcal{T}, \alpha), [\])$. Let C be a list of dual cycles of (\mathcal{T}, α) . Then $\text{Specialization}((\mathcal{T}, \alpha), P, C)$ is equal to $\rho_C(P)$.

Proof. The whole proof follows from the discussion on page 10. Let n be the number of tetrahedra of \mathcal{T} . Let Γ be the dual graph of \mathcal{T} . The pair (U, r) on line 2 of the algorithm is such that r equals the rank of H_M and the last r columns of the inverse $U^{-1} \in \text{GL}(n+1, \mathbb{Z})$ give the expressions for the basis elements of H_M as simplicial 1-cycles in Γ_Υ , the graph obtained from Γ by contracting a spanning tree Υ to a point. The pair (U', s) on line 4 of the algorithm is such that s equals the rank of H_M^C and the last s rows of the matrix $U' \in \text{GL}(n+1, \mathbb{Z})$ encode the H_M^C -pairings of the faces dual to the edges of Γ_Υ .

Let A_1 be the matrix obtained from U^{-1} by deleting its first $n + 1 - r$ columns. Let A_2 be the matrix obtained from U' by deleting its first $n + 1 - s$ rows. Then the matrix $A = A_2 \cdot A_1$ represents the epimorphism $\rho_C : H_M \rightarrow H_M^C$ written in terms of the bases of H_M, H_M^C fixed by the algorithm FacePairings . Note that we use the fact that the algorithm FacePairings is deterministic; see Remark 4.7.

Each monomial $a_h \cdot h$ in P can be encoded by a pair (a_h, v) where $a_h \in \mathbb{Z}, v \in \mathbb{Z}^r$. Then the pair $(a_h, A_2 A_1(v))$ encodes the corresponding monomial $a_h \cdot \rho_C(h)$ appearing in $\rho_C(P)$. Therefore the polynomial spec on line 12 of the algorithm is equal to $\rho_C(P)$. \square

Note that it only make sense to apply the algorithm Specialization to an element of $\mathbb{Z}[H_M]$ which, as a Laurent polynomial, is expressed in terms of the basis fixed by the algorithm FacePairings .

8.4. Algorithm Teichmüller Polynomial

Let S be an oriented, closed or punctured, surface of finite genus. Let $\psi : S \rightarrow S$ be a pseudo-Anosov homeomorphism. Denote by N the mapping torus of ψ . Let F be the fibered face of the Thurston norm on $H_2(N, \partial N; \mathbb{R})$ such that $[S] \in \mathbb{R}_+ \cdot F$. We give an algorithm which computes the Teichmüller polynomial of F .

By **Veering** we denote an algorithm which given a pseudo-Anosov homeomorphism $\psi : S \rightarrow S$ outputs

- the veering triangulation \mathcal{V} of the mapping torus of $\hat{\psi} : \hat{S} \rightarrow \hat{S}$, where \hat{S} is obtained from S by puncturing it at the singularities of ψ and $\hat{\psi}$ is the restriction of ψ to \hat{S} ,
- a nonnegative solution $w = (w_1, \dots, w_{2n})$ to the system of branch equations of \mathcal{V} such that the carried surface $S^w = \sum_{i=1}^{2n} w_i f_i$ is homologous to the fiber \hat{S} .

Algorithm **Veering** is explained in [1, Section 4]. It has been implemented by Mark Bell in `flipper` [3].

Algorithm TeichmüllerPolynomial

Computing the Teichmüller polynomial of a fibered face of the Thurston norm ball

Input: A pseudo-Anosov homeomorphism $\psi : S \rightarrow S$ **Output:** The Teichmüller polynomial of the face F in $H_2(N, \partial N; \mathbb{R})$, where N is the mapping torus of ψ and $[S] \in \mathbb{R}_+ \cdot F$

- 1: $(\mathcal{V}, w) := \text{Veering}(\psi)$
- 2: $C := \text{BoundaryCycles}(\mathcal{V}, w)$
- 3: if the j -th torus cusp is not filled in N , remove $C[j]$ from the list C
- 4: $\Theta := \text{Specialization}(\mathcal{V}, \text{TautPolynomial}(\mathcal{V}), C)$
- 5: **return** Θ

Proposition 8.6. Let $\psi : S \rightarrow S$ be a pseudo-Anosov homeomorphism. Denote by N its mapping torus. Let F be the fibered face of the Thurston norm ball in $H_2(N, \partial N; \mathbb{Z})$ such that $[S] \in \mathbb{R}_+ \cdot F$. Then $\text{TeichmüllerPolynomial}(\psi)$ is equal to the Teichmüller polynomial Θ_F of F .

Proof. Let \hat{S} denote the surface obtained from S by puncturing it at the singularities of the invariant foliations of ψ . The pair (\mathcal{V}, w) on line 1 consists of the veering triangulation of $M = N - \text{sing}(F)$ associated to F , and a nonnegative solution to its system of branch equations which puts \hat{S} in a fixed carried position. Then the list C constructed on line 2 consists of dual cycles homologous to the boundary components of $\hat{S} = S^w$. If a boundary torus T of the underlying manifold of \mathcal{V} is not filled in N , we remove the dual cycle encoding $S^w \cap T$ from the list C . After this, C satisfies $H_N = H_M^C$, where the meaning of the superscript C is as in (4.3).

Let $i_* : H_M \rightarrow H_N$ be the epimorphism determined by the inclusion of M into N . Since $i_* = \rho_C$, by Proposition 8.5 the polynomial Θ on line 4 is equal to $i_*(\Theta_{\mathcal{V}})$ and hence, by Theorem 8.1, to Θ_F . \square

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Declaration of Interest

No potential conflict of interest was reported by the author(s).

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